

ROMANIAN MATHEMATICAL MAGAZINE

Let $A = \{1^\circ, 2^\circ, \dots, 89^\circ\}$. Let $B = \{x_1^\circ, x_2^\circ, \dots, x_{89}^\circ\}$ be a permutation of the elements of A , where $x_k \in A$, for all $k \in \{1, 2, \dots, 89\}$
 Prove that the following sum S satisfies the inequality :

$$S = \sum_{k=1}^{89} \frac{\sin^{2026}(k^\circ)}{\cos^{2025}(x_k^\circ)} > 56$$

Proposed by Gheorghe Crăciun-Romania

Solution 1 by Khaled Abd Imouti, Kasem Abotrabai-Syria

$$S = \sum_{k=1}^{89} \frac{\sin^{2026}(k^\circ)}{\cos^{2025}(x_k^\circ)}$$

by using Radon's inequality:

$$S \geq \frac{(\sin_1^\circ + \sin_2^\circ + \dots + \sin_{89}^\circ)}{(\cos_1^\circ + \cos_2^\circ + \dots + \cos_{89}^\circ)}$$

$$S \geq (\sin_1^\circ + \sin_2^\circ + \dots + \sin_{89}^\circ) \Rightarrow$$

$$2\sin \frac{1^\circ}{2} \cdot S \geq (2\sin_1^\circ \cdot \sin \frac{1^\circ}{2}) + (2\sin_2^\circ \cdot \sin \frac{1^\circ}{2}) + \dots + (2\sin_{89}^\circ \cdot \sin \frac{1^\circ}{2})$$

$$2\sin \frac{1^\circ}{2} \cdot S \geq (\cos \frac{1^\circ}{2} - \sin \frac{3^\circ}{2}) + (\cos \frac{3^\circ}{2} - \sin \frac{5^\circ}{2}) + \dots + (\cos \frac{177^\circ}{2} - \sin \frac{179^\circ}{2})$$

$$2\sin \frac{1^\circ}{2} \cdot S \geq \cos \frac{1^\circ}{2} - \sin \frac{179^\circ}{2} \Rightarrow S \geq \frac{\cos \frac{1^\circ}{2} - \sin \frac{179^\circ}{2}}{2\sin \frac{1^\circ}{2}} \Rightarrow S \geq \frac{\cos \frac{1^\circ}{2} - \sin \frac{179^\circ}{2}}{2\sin \frac{1^\circ}{2}} \Rightarrow$$

$$S \geq \frac{2 \sin 45^\circ \cdot \sin \frac{89^\circ}{2}}{2\sin \frac{1^\circ}{2}} \Rightarrow S \geq \frac{0.707 \times 0.700}{0.008} \Rightarrow S \geq 56.88 \dots \Rightarrow S > 56$$

Solution 2 by Khaled Abd Imouti, Kasem Abotrabai-Syria

$$S = \sum_{k=1}^{89} \frac{\sin^{2026}(k^\circ)}{\cos^{2025}(x_k^\circ)}$$

by using Radon's inequality:

$$S \geq \frac{(\sin_1^\circ + \sin_2^\circ + \dots + \sin_{89}^\circ)}{(\cos_1^\circ + \cos_2^\circ + \dots + \cos_{89}^\circ)}$$

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$S \geq \sin_1^\circ + \sin_2^\circ + \dots + \sin_{89}^\circ \Rightarrow$ Lagrange and Euler

$$\sum_{k=1}^n \sin(k\theta) = \frac{\sin \frac{n\theta}{2} \cdot \sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}, \theta = 1^\circ \Rightarrow$$

$$S \geq \frac{\sin \frac{89^\circ}{2} \cdot \sin \frac{90^\circ}{2}}{\sin \frac{1^\circ}{2}} \Rightarrow S \geq \frac{\sin \frac{89^\circ}{2} \cdot \sin 45^\circ}{\sin \frac{1^\circ}{2}} \Rightarrow$$

$$S \geq (0.707 \times 0.7009) \div 0.008 \Rightarrow$$

$$S \geq 0.707 \times 80.319 \Rightarrow S \geq 56.785 \Rightarrow S > 56$$