ROMANIAN MATHEMATICAL MAGAZINE

$$If\ \ 0 < x, y < \frac{\pi}{2}\ \ then:$$

$$\frac{cos^4(x)}{sin^3(x)} + \frac{sin^4(x)}{cos^3(y)} + \frac{cos^4(y)}{sin^3(y)} + \frac{sin^4(y)}{cos^3(x)} \geq \frac{8\sin(2x)\sin(2y)}{\sin(2x)\,(\sin(y)+\cos(y)) + \sin(2y)\,(\sin(x)+\cos(x))}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$Let\ \ \sin(x) = a, \cos(x) = b > 0\ \ , \sin(y) = c > 0, \cos(y) = d > 0$$

$$We\ \ will\ \ obtain\ the\ expression\ \ equivalent\ to\ the\ given\ inequality$$

$$\frac{b^4}{a^3} + \frac{a^4}{d^3} + \frac{d^4}{c^3} + \frac{c^4}{b^3} \geq \frac{32abcd}{2ab(c+d) + 2cd(a+b)}$$

$$LHS = \frac{b^4}{a^3} + \frac{a^4}{d^3} + \frac{d^4}{c^3} + \frac{c^4}{b^3} \overset{Radon}{\geq} \frac{(a+b+c+d)^4}{(a+b+c+d)^3} = a+b+c+d$$

$$We\ must\ show:$$

$$a+b+c+d \geq \frac{32abcd}{2ab(c+d) + 2cd(a+b)}$$

$$(a+b+c+d)\big(ab(c+d) + cd(a+b)\big) \geq 16abcd$$

$$(a+b+c+d)\big(ab(c+d) + cd(a+b)\big) \overset{A-G}{\geq} 4\sqrt[4]{abcd}\big(ab.2\sqrt{cd} + cd.2\sqrt{ab}\big) \overset{A-G}{\geq}$$

$$\geq 8\sqrt[4]{abcd}\left(2\sqrt{abcd\sqrt{abcd}}\right) = 16\sqrt[4]{abcd}\,.\sqrt{abcd}\,.\sqrt[4]{abcd} = 16abcd$$

$$Equality\ \ holds\ \ for:\ x = y = \frac{\pi}{4}.$$