ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$, then :

$$(xy + y - 1)\left(x + \frac{1}{y} - 1\right)\left(\frac{1}{x} + \frac{1}{xy} - 1\right) \le 1$$

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$$(xy + y - 1)\left(x + \frac{1}{y} - 1\right)\left(\frac{1}{x} + \frac{1}{xy} - 1\right) \overset{?}{\le} 1$$

$$\Leftrightarrow \Big(\overbrace{xy + y - 1}^{\sigma_1}\Big)\Big(\overbrace{xy + 1 - y}^{\sigma_2}\Big)\Big(\overbrace{y + 1 - xy}^{\sigma_3}\Big) \boxed{\underset{(*)}{\overset{?}{\le}}} xy^2$$

Now, if all of σ_1, σ_2 and $\sigma_3 \le 0$ or if exactly one among σ_1, σ_2 and $\sigma_3 \le 0$, then $(*)$ is trivially true as $xy^2 > 0$

Now, if possible, let us assume exactly 2 among σ_1, σ_2 and $\sigma_3 \le 0$ and then, 3 cases arise :

Case 1 $\sigma_1, \sigma_2 \le 0$ (and $\sigma_3 > 0$) and then : $\sigma_1 + \sigma_2 \le 0$

$\Rightarrow xy + y - 1 + xy + 1 - y \le 0 \Rightarrow xy \le 0 \to$ impossible since $x, y > 0 \Rightarrow xy > 0$

Case 2 $\sigma_2, \sigma_3 \le 0$ (and $\sigma_1 > 0$) and then : $\sigma_2 + \sigma_3 \le 0$

$\Rightarrow xy + 1 - y + y + 1 - xy \le 0 \Rightarrow 2 \le 0 \to$ impossible

Case 3 $\sigma_3, \sigma_1 \le 0$ (and $\sigma_2 > 0$) and then : $\sigma_3 + \sigma_1 \le 0$

$\Rightarrow y + 1 - xy + xy + y - 1 \le 0 \Rightarrow y \le 0 \to$ impossible as $y > 0$

Hence, our assumption is incorrect and we conclude that : exactly 2 among σ_1, σ_2 and σ_3 cannot be ≤ 0

So, only one scenario remains, which is : $\sigma_1, \sigma_2, \sigma_3 > 0$ and then :

$$(*) \Leftrightarrow \sigma_1^2\sigma_2^2\sigma_3^2 \boxed{\underset{(**)}{\overset{?}{\le}}} x^2y^4 \text{ and we have : } \sigma_1^2\sigma_2^2\sigma_3^2 =$$

$$(xy + y - 1)(xy + 1 - y).(xy + 1 - y)(y + 1 - xy).(y + 1 - xy)(xy + y - 1)$$

$$= (x^2y^2 - (y-1)^2)(1 - (xy - y)^2)(y^2 - (xy - 1)^2)$$

$$\boxed{\le} x^2y^2(1 - (xy - y)^2)(y^2 - (xy - 1)^2)$$

$$\left(\because \sigma_1, \sigma_2, \sigma_3 > 0 \Rightarrow \sigma_2\sigma_3.\sigma_3\sigma_1 > 0 \Rightarrow (1 - (xy - y)^2)(y^2 - (xy - 1)^2) > 0 \text{ and } \because -(y-1)^2 \le 0\right)$$

$$\boxed{\le} x^2y^2(y^2 - (xy - 1)^2)$$

$$(\because \sigma_1, \sigma_2, \sigma_3 > 0 \Rightarrow \sigma_3\sigma_1 > 0 \Rightarrow y^2 - (xy - 1)^2 > 0 \text{ and } \because -(xy - y)^2 \le 0)$$

$\boxed{\le} x^2y^4$ $(\because -(xy - 1)^2 \le 0) \Rightarrow (**) \Rightarrow (*)$ is true whenever $\sigma_1, \sigma_2, \sigma_3 > 0$ and combining all possible scenarios regarding the signs of σ_1, σ_2 and σ_3,

we conclude : $(xy + y - 1)\left(x + \frac{1}{y} - 1\right)\left(\frac{1}{x} + \frac{1}{xy} - 1\right) \le 1 \; \forall \, x, y > 0$,

$'' = ''$ iff $y = 1 \wedge xy = y \wedge xy = 1 \Rightarrow '' = ''$ iff $x = y = 1$ (QED)