A NEW MEHMET ŞAHIN'S CONFIGURATION

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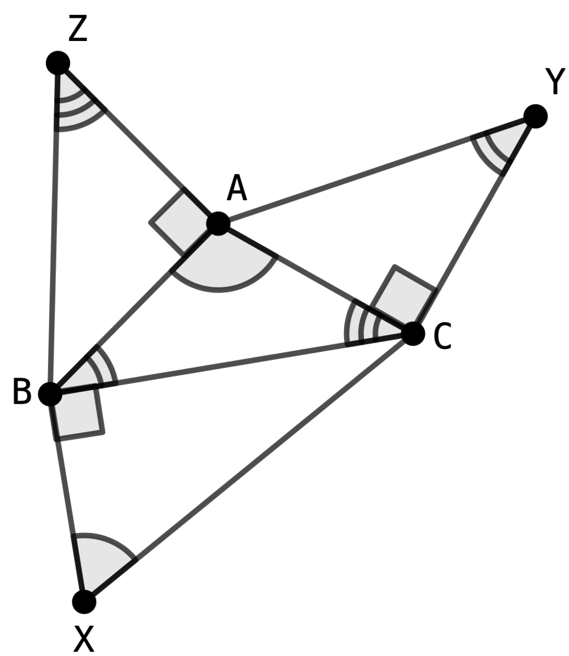
ABSTRACT. Let be the triangle ABC with sides a, b, c. Let be the points $X, Y, Z \notin Int(\Delta ABC)$ such that $m(\angle BAZ) = m(\angle CBX) = m(\angle ACY) = 90°$ and $m(\angle BXC) = m(\angle A); m(\angle CYA) = m(\angle B); m(\angle AZB) = m(\angle C)$. In this article we will study the properties of this configuration.



Property 1.

$$[BXC] + [CYA] + [AZB] = 2[ABC]$$

Proof.
Lemma.
In any triangle ABC with area F holds:

$$16F^2 = 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4$$

Proof of lemma.
By Heron's formula:

$$F^2 = s(s-a)(s-b)(s-c)$$

$$F^2=\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}\cdot\frac{a+c-b}{2}\cdot\frac{a+b-c}{2}$$
$$16F^2=((b+c)^2-a^2)(a^2-(b-c)^2)$$
$$16F^2=(b^2+c^2-a^2+2bc)(a^2-b^2-c^2+2bc)$$
$$16F^2=4b^2c^2-(b^2+c^2-a^2)^2$$
$$16F^2=4b^2c^2-b^4-c^4-a^4+2a^2b^2+2b^2c^2-2b^2c^2$$
$$16F^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$$
$$16F^2=2\sum_{cyc}a^2b^2-\sum_{cyc}a^4 \tag{1}$$

□

Back to the main proof:
In ΔCBX:
$$\cot A=\frac{BX}{BC}\Rightarrow BX=BC\cot A=a\cot A$$
Analogous:
$$CY=b\cot B; AZ=c\cot C$$
$$[BXC]+[CYA]+[AZB]=$$
$$=\frac{BC\cdot BX}{2}+\frac{CA\cdot CY}{2}+\frac{AB\cdot AZ}{2}=$$
$$=\frac{a^2\cdot\cot A}{2}+\frac{b^2\cot B}{2}+\frac{c^2\cot C}{2}=$$
$$=\sum_{cyc}\frac{a^2\cot A}{2}=\frac{1}{2}\sum_{cyc}\frac{a^2\cos A}{\sin A}=$$
$$=\frac{1}{2}\cdot\sum_{cyc}\frac{a^2(b^2+c^2-a^2)}{2bc\sin A}=\frac{1}{2}\sum_{cyc}\frac{a^2(b^2+c^2-a^2)}{4F}=$$
$$=\frac{1}{8F}\sum_{cyc}(a^2b^2+a^2c^2-a^4)=$$
$$=\frac{1}{8F}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)\overset{(1)}{=}$$
$$=\frac{1}{8F}\cdot 16F^2=2F=2[ABC]$$

□

Property 2.
$$AZ+ZB+BX+XC+CY+YA=2(r_a+r_b+r_c)$$

Proof.
Lemma.
In any triangle ABC with r - inradii; R - circumradii, r_a, r_b, r_c - exradies holds:
$$r_a+r_b+r_c=4R+r \tag{2}$$

Proof of lemma.

$$r_a + r_b + r_c = \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} =$$

$$= F\sum_{cyc}\frac{1}{s-a} = \frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}(s-b)(s-c) =$$

$$= \frac{Fs}{s(s-a)(s-b)(s-c)}\cdot\sum_{cyc}(s^2 - s(b+c) + bc) =$$

$$= \frac{Fs}{F^2}\sum_{cyc}(s^2 - s(2s-a) + bc) =$$

$$= \frac{s}{F}\sum_{cyc}(s^2 - 2s^2 + sa + bc) =$$

$$= \frac{s}{rs}\left(-3s^2 + s\sum_{cyc}a + \sum_{cyc}bc\right) =$$

$$= \frac{1}{r}(-3s^2 + s\cdot 2s + s^2 + r^2 + 4Rr) =$$

$$= \frac{1}{r}(4Rr + r^2) = 4R + r$$

□

Back to the main proof:
In ΔBXC:

$$XC^2 = BX^2 + BC^2 = a^2\cot^2 A + a^2 = a^2(1 + \cot^2 A) =$$

$$= a^2\Big(1 + \frac{\cos^2 A}{\sin^2 A}\Big) = a^2\cdot\frac{\sin^2 A + \cos^2 A}{\sin^2 A} = \frac{a^2}{\sin^2 A}$$

$$XC = \frac{a}{\sin A} = \frac{2R\sin A}{\sin A} = 2R$$

Analogous: $YA = ZB = 2R$

$$AZ + ZB + BX + XC + CY + YA =$$

$$= c\cot C + 2R + a\cot A + 2R + b\cot B + 2R =$$

$$= 6R + \sum_{cyc}a\cot A = 6R + \sum_{cyc}a\cdot\frac{\cos A}{\sin A} =$$

$$= 6R + \sum_{cyc}\frac{a}{\sin A}\cdot\cos A = 6R + \sum_{cyc}2R\cos A =$$

$$= 6R + 2R\sum_{cyc}\cos A = 6R + 2R\Big(1 + \frac{r}{R}\Big) =$$

$$= 6R + 2R + 2r = 8R + 2r = 2(4R + r) =$$

$$\overset{(2)}{=} 2(r_a + r_b + r_c)$$

□

Property 3.

$$[AZBXCY] = 3[ABC]$$

Proof.
Lemma: In ΔABC the following relationship holds:

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C \tag{3}$$

Proof of lemma.

$$\begin{gathered}
\sin 2A + \sin 2B + \sin 2C = \\
= 2 \sin \frac{2A+2B}{2} \cos \frac{2A-2B}{2} + \sin 2C = \\
= 2 \sin(A+B) \cos(A-B) + \sin 2C = \\
= 2 \sin(\pi - C) \cos(A-B) + \sin 2C = \\
= 2 \sin(\cos(A-B) + 2 \sin C \cos C = \\
= 2 \sin C(\cos(A-B) + \cos C) = \\
= 2 \sin C \cdot 2 \cos \frac{A-B+C}{2} \cos \frac{A-B-C}{2} = \\
= 4 \sin C \cos \frac{\pi - 2B}{2} \cos \frac{A-(\pi-A)}{2} = \\
= 4 \sin C \cos\left(\frac{\pi}{2} - B\right) \cos\left(A - \frac{\pi}{2}\right) = \\
= 4 \sin C \sin B \sin A
\end{gathered}$$

□

Back to the main proof:

$$\begin{gathered}
[AZBXCY] = \\
= [ABC] + [ABZ] + [BCX] + [CAY] = \\
= F + \sum_{cyc} \frac{AB \cdot AZ}{2} = F + \sum_{cyc} \frac{c \cdot \cot C}{2} = \\
= F + \frac{1}{2} \sum_{cyc} c^2 \cdot \frac{\cos C}{\sin C} = \\
= F + \frac{1}{2} \sum_{cyc} \frac{c}{\sin C} \cdot c \cos C = \\
= F + \frac{1}{2} \sum_{cyc} 2R \cdot 2R \sin C \cos C = \\
= F + \frac{1}{2} \cdot 2R^2 \sum_{cyc} 2 \sin C \cos C = \\
= F + R^2 \sum_{cyc} \sin 2C \overset{(3)}{=} \\
= F + R^2 \cdot 4 \sin A \sin B \sin C = \\
= F + 4R^2 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \\
= F + \frac{abc}{2R} = F + \frac{4RF}{2R} = \\
= F + 2F = 3F = 3[ABC]
\end{gathered}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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