ROMANIAN MATHEMATICAL MAGAZINE

If in Δ ABC the following relationship holds : $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$

then prove that : $2\left(\frac{R}{r}\right)\le\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)\le\left(\frac{R}{r}\right)^2$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}\Rightarrow a.\frac{s(s-c)}{ab}+c.\frac{s(s-a)}{bc}=\frac{3b}{2}$$
$$\Rightarrow\frac{(a+b)^2-c^2+(b+c)^2-a^2}{4b}=\frac{3b}{2}$$
$$\Rightarrow a^2+b^2+2ab-c^2+b^2+c^2+2bc-a^2=6b^2$$
$$\Rightarrow 4b^2=2b(c+a)\Rightarrow 2b=c+a\Rightarrow 8R\sin\frac{B}{2}\cos\frac{B}{2}=4R\cos\frac{B}{2}\cos\frac{C-A}{2}$$
$$\Rightarrow\boxed{\cos\frac{C-A}{2}\overset{(*)}{=}2\sin\frac{B}{2}}\Rightarrow 2\sin\frac{B}{2}\le 1\ \left(\because\cos\frac{C-A}{2}\le 1\right)$$
$$\Rightarrow\sin\frac{B}{2}\le\frac{1}{2}\Rightarrow\frac{B}{2}\le\frac{\pi}{6}\Rightarrow\boxed{B\overset{(**)}{\le}\frac{\pi}{3}}$$
$$\text{Now, }\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)=\frac{1}{s}.(r_a+r_c).s.\frac{r_a+r_c}{r_ar_c}=$$
$$\frac{4R\cos^2\frac{B}{2}.4R\cos^2\frac{B}{2}}{s(s-b)}=\frac{4Rs(s-b)}{cas(s-b)}.4R\cos^2\frac{B}{2}=\frac{16R^2\cos^2\frac{B}{2}}{16R^2\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}}\le\left(\frac{R}{r}\right)^2$$
$$\Leftrightarrow\frac{\cos^2\frac{B}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}}\le\frac{1}{16\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}$$
$$\Leftrightarrow 16\cos^2\frac{B}{2}\sin^2\frac{B}{2}.\left(2\sin\frac{A}{2}\sin\frac{C}{2}\right)\le 2\cos\frac{C}{2}\cos\frac{A}{2}$$
$$\Leftrightarrow(4\sin^2 B)\left(\cos\frac{C-A}{2}-\sin\frac{B}{2}\right)\le\sin\frac{B}{2}+\cos\frac{C-A}{2}$$
$$\overset{\text{via }(*)}{\Leftrightarrow}(4\sin^2 B)\left(2\sin\frac{B}{2}-\sin\frac{B}{2}\right)\le\sin\frac{B}{2}+2\sin\frac{B}{2}\Leftrightarrow\boxed{\boxed{\sin^2 B\le\frac{3}{4}}}$$
$$\rightarrow\text{true}\because B\overset{\text{via }(**)}{\le}\frac{\pi}{3}\Rightarrow\sin B\le\frac{\sqrt{3}}{2}\Rightarrow\sin^2 B\le\frac{3}{4}$$
$$\therefore\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)\le\left(\frac{R}{r}\right)^2$$
$$\text{Again, }\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right)\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)\ge 2\left(\frac{R}{r}\right)$$
$$\Leftrightarrow\frac{\cos^2\frac{B}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}\cos\frac{A}{2}\sin\frac{A}{2}}\ge\frac{1}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\Leftrightarrow 4\sin\frac{B}{2}\cos^2\frac{B}{2}\ge 2\cos\frac{C}{2}\cos\frac{A}{2}$$

$$= \sin\frac{B}{2} + \cos\frac{C-A}{2} \overset{\text{via }(*)}{=} 3\sin\frac{B}{2} \Leftrightarrow \boxed{\cos^2\frac{B}{2} \geq \frac{3}{4}} \to \text{true} \because \frac{B}{2} \overset{\text{via }(**)}{\leq} \frac{\pi}{6}$$

$$\Rightarrow \cos^2\frac{B}{2} \geq \cos^2\frac{\pi}{6} = \frac{3}{4} \therefore \left(\tan\frac{A}{2} + \tan\frac{C}{2}\right)\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) \geq 2\left(\frac{R}{r}\right) \textit{ and so,}$$

$$2\left(\frac{R}{r}\right) \leq \left(\tan\frac{A}{2} + \tan\frac{C}{2}\right)\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) \leq \left(\frac{R}{r}\right)^2$$

whenever $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\frac{3b}{2} = a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = a.\frac{s(s-c)}{ab} + c.\frac{s(s-a)}{bc} = s$, then $2b = a + c$

$$\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right)\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) = \left(\frac{r}{s-a} + \frac{r}{s-c}\right)\left(\frac{s-a}{r} + \frac{s-c}{r}\right) = \frac{b^2}{(s-a)(s-c)} =$$

$$= \frac{abc}{(s-a)(s-b)(s-c)}.\frac{b(c+a-b)}{2ca} = \frac{4Rsr}{sr^2}.\frac{b^2}{2ca} = 2\frac{R}{r}.\frac{(a+c)^2}{4ca} \geq 2\frac{R}{r}$$

$$\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right)\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) = \frac{b^2}{(s-a)(s-c)} =$$

$$= \left(\frac{abc}{(s-a)(s-b)(s-c)}\right)^2.\frac{(s-a)(s-b).(s-b)(s-c)}{a^2c^2} \leq$$

$$\leq \left(\frac{4R}{r}\right)^2.\frac{[(s-a)+(s-b)]^2.[(s-b)+(s-c)]^2}{16a^2c^2} = \left(\frac{R}{r}\right)^2.$$

which completes the proof. Equality holds iff ΔABC is equilateral.