ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a^2+b^2+c^2}{2R}\le\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a\le\frac{9R}{2}$$

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$$\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a=\sum_{cyc}\left(\frac{2bc}{b^2+c^2}.\frac{m_am_b}{\left(\frac{bc}{2R}\right)}\right)=4R\sum_{cyc}\frac{m_am_b}{b^2+c^2}$$

$$\therefore\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a\le\frac{9R}{2}\Leftrightarrow\sum_{cyc}\frac{m_am_b}{b^2+c^2}\le\frac{9}{8}\to (1)$$

We shall now prove that : $\sum_{cyc}\frac{ab}{m_b^2+m_c^2}\le 2$ and indeed, $\sum_{cyc}\frac{ab}{m_b^2+m_c^2}=$

$$4.\sum_{cyc}\frac{ab}{\sum_{cyc}a^2+3a^2}\overset{A-G}{\le}4.\sum_{cyc}\frac{ab}{2a.\sqrt{3\sum_{cyc}a^2}}=\frac{2}{\sqrt{3\sum_{cyc}a^2}}.\sum_{cyc}b=2.\frac{\sum_{cyc}a}{\sqrt{3\sum_{cyc}a^2}}$$

≤ 2 and implementing $\sum_{cyc}\frac{ab}{m_b^2+m_c^2}\le 2$ on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$

whose medians as a consequence of trivial calculations $=\frac{a}{2},\frac{b}{2},\frac{c}{2}$, we get :

$$\sum_{cyc}\frac{\frac{4}{9}m_am_b}{\left(\frac{b^2+c^2}{4}\right)}\le 2\Rightarrow\sum_{cyc}\frac{m_am_b}{b^2+c^2}\le\frac{9}{8}\Rightarrow (1)\text{ is true }\therefore\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a\le\frac{9R}{2}$$

$$\text{and }\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a\ge\sum_{cyc}m_a\overset{Tereshin}{\ge}\sum_{cyc}\frac{b^2+c^2}{4R}=\frac{a^2+b^2+c^2}{2R}$$

$$\text{and so, }\frac{a^2+b^2+c^2}{2R}\le\frac{s_a}{h_a}.m_b+\frac{s_b}{h_b}.m_c+\frac{s_c}{h_c}.m_a\le\frac{9R}{2}\ \forall\,\Delta\text{ ABC},$$

'' = '' iff Δ ABC is equilateral (QED)