ROMANIAN MATHEMATICAL MAGAZINE

If in ΔABC : $x = 2R\sum_{cyc}\frac{1}{w_a}cos\frac{B-C}{2}$ and $y = \frac{4}{x}$, then prove that :

$$\frac{1}{h_a}\left(\frac{h_a}{h_b}\right)^y + \frac{1}{h_b}\left(\frac{h_b}{h_c}\right)^y + \frac{1}{h_c}\left(\frac{h_c}{h_a}\right)^y \le \frac{1}{r}$$

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We have :

$$x = 2R\sum_{cyc}\frac{1}{w_a}cos\frac{B-C}{2} = 2R\sum_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}\cdot\frac{b+c}{4R\cos\frac{A}{2}} =$$

$$= \frac{1}{4}\sum_{cyc}\frac{(b+c)^2}{s(s-a)} = \frac{1}{4s}\sum_{cyc}\frac{\big(a+2(s-a)\big)^2}{s-a} \ge$$

$$\overset{AM-GM}{\ge} \frac{1}{4s}\sum_{cyc}\frac{4.2(s-a)a}{s-a} = 4 \ \Rightarrow \ y \le 1.$$

By Bernoulli's inequality, we have :

$$\sum_{cyc}\frac{1}{h_a}\left(\frac{h_a}{h_b}\right)^y = \sum_{cyc}\frac{1}{h_a}\left(1+\left(\frac{h_a}{h_b}-1\right)\right)^y \le \sum_{cyc}\frac{1}{h_a}\left(1+y\left(\frac{h_a}{h_b}-1\right)\right) = \sum_{cyc}\frac{1}{h_a} = \frac{1}{r}$$

Equality holds iff ΔABC is equilateral.