ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{2}{3}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\geq\sqrt[5]{\frac{12sr}{R^2}}$$

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Solution by Tapas Das-India

$$\frac{2}{3}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\overset{AM-GM}{\geq}\frac{2}{3}\,.\,3\sqrt[3]{\left(\cos\frac{A}{2}\,.\cos\frac{B}{2}\,.\cos\frac{C}{2}\right)}=2\sqrt[3]{\frac{s}{4R}}$$

$$We\ need\ to\ show\ 2\sqrt[3]{\frac{s}{4R}}\geq\sqrt[5]{\frac{12sr}{R^2}}\ \ or\ \left(2\sqrt[3]{\frac{s}{4R}}\right)^{15}\geq\left(\sqrt[5]{\frac{12sr}{R^2}}\right)^{15}$$

$$2^{15}\frac{s^5}{2^{10}R^5}\geq\frac{12^3s^3r^3}{R^6}$$

$$2^{15}\frac{s^5}{2^{10}R^5}\geq\frac{3^32^6s^3r^3}{R^6}$$

$$Rs^2\geq 54r^3$$

$$R\ \cdot 27\,r^2\overset{Mitrinovic}{\geq}54r^3$$

$$R\geq 2r\ \ (Euler)$$

$$Equality\ holds\ for\ \ an\ equilateral\ triangle.$$