ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, prove that :

$$\left|\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\right| \le \frac{1}{24\sqrt{3}}\cdot\frac{\sqrt{(p^2-12Rr-3r^2)^3}}{R^2r}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Mollweide's formula, we have $\sin\frac{B-C}{2} = \frac{b-c}{a}.\cos\frac{A}{2}$ (and analogs), then:

$$\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2} = \frac{(a-b)(b-c)(c-a)}{4pRr}.\frac{p}{4R} = \frac{(a-b)(b-c)(c-a)}{16R^2r},$$

and since $(a-b)^2+(b-c)^2+(c-a)^2 = 2\left(p^2-12Rr-3r^2\right)$,

then the desired inequality is equivalent to

$$3\sqrt{6}.|(a-b)(b-c)(c-a)| \le \sqrt{[(a-b)^2+(b-c)^2+(c-a)^2]^3}.$$

WLOG, we assume that $a \ge b \ge c$.

Let $x := a-b, y := b-c$. The desired inequality becomes

$$3\sqrt{3}xy(x+y) \le 2\sqrt{(x^2+y^2+xy)^3}$$

By AM − GM inequality, we have

$$3\sqrt{3}xy(x+y) = 2\sqrt{27.xy.xy.\frac{(x+y)^2}{4}} \le 2\sqrt{\left(xy+xy+\frac{(x+y)^2}{4}\right)^3} \le 2\sqrt{(x^2+y^2+xy)^3}$$

So the proof is complete. Equality holds iff $x = y \Leftrightarrow a+c = 2b$ and permutation.