ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2) \le \frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}$$

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$$\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2) \le \frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}$$

$$\Leftrightarrow \frac{2(s^2-4Rr-r^2)}{4r^2s^2}.\left((4R+r)^2-2s^2\right) \le$$

$$\le \frac{\left(4R+r-\sqrt{2r(4R+r)}\right)\left(4R+r-\sqrt{2r(4R+r)}\right)}{2r^2}$$

$$\Leftrightarrow \frac{s^2-4Rr-r^2}{s^2}.\left((4R+r)^2-2s^2\right) \le (4R+r)^2+2r(4R+r)-2(4R+r).\sqrt{2r(4R+r)}$$

$$\Leftrightarrow (4R+r)^2-2s^2-\frac{(4Rr+r^2)(4R+r)^2}{s^2}+2(4Rr+r^2) \le$$

$$\le (4R+r)^2+2(4Rr+r^2)-2(4R+r).\sqrt{2(4Rr+r^2)}$$

$$\Leftrightarrow 2s^4+(4Rr+r^2)(4R+r)^2-2s^2(4R+r).\sqrt{2(4Rr+r^2)} \ge 0$$

$$\Leftrightarrow \left(\sqrt{2}s^2\right)^2+\left(\sqrt{4Rr+r^2}.(4R+r)\right)^2-2\left(\sqrt{2}s^2\right).\left(\sqrt{4Rr+r^2}.(4R+r)\right) \ge 0$$

$$\Leftrightarrow \left(\sqrt{2}s^2-\sqrt{4Rr+r^2}.(4R+r)\right)^2 \ge 0 \rightarrow \text{true}$$

$$\therefore \left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)(r_a^2+r_b^2+r_c^2) \le$$

$$\le \frac{\sqrt{r+4R}\left(r+4R-\sqrt{2r(r+4R)}\right)\left(\sqrt{r+4R}-\sqrt{2r}\right)}{2r^2}\ \forall\,\Delta\,\text{ABC (QED)}$$