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In any ΔABC prove that :

$$\frac{1}{\sqrt{r_a}}+\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\le\frac{\sqrt{6}}{9}\cdot\frac{w_a+w_b+w_c}{r\sqrt{R}}$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$w_a=\frac{2\sqrt{bcs(s-a)}}{b+c}=\sqrt{\frac{8abc.s(s-a)}{2a(b+c)^2}}\overset{AM-GM}{\geq}\sqrt{\frac{27.8abc.s(s-a)}{[2a+(b+c)+(b+c)]^3}}=$$

$$=\frac{9}{\sqrt{6}}\cdot\sqrt{\frac{Rr(s-a)}{s}}=\frac{9r\sqrt{R}}{\sqrt{6}}\cdot\frac{1}{\sqrt{r_a}}\ \Rightarrow\ \frac{\sqrt{6}}{9}\cdot\frac{w_a}{r\sqrt{R}}\geq\frac{1}{\sqrt{r_a}}\ \ \text{(and analogs)}$$

Adding this inequality with similar ones yields the desired result.

Equality holds iff ΔABC is equilateral.