ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship ho$l$ds :

$$3 \le \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10}$$

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$$845 > 841 \Rightarrow \sqrt{5*169} > 29 \Rightarrow \frac{\sqrt{5}}{10} > \frac{29}{130} = \frac{12}{13} - \frac{7}{10} \Rightarrow \frac{7+\sqrt{5}}{10} > \frac{12}{13}$$

$$\Rightarrow 3+\frac{7+\sqrt{5}}{10} > \frac{51}{13} \overset{?}{>} \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$$

$$\Leftrightarrow \frac{51}{13} \overset{?}{>} \frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left((a+b)\left(\sum_{cyc}ab+a^2\right)\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)}\cdot\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}(a+b)\right)+\sum_{cyc}a^3+\sum_{cyc}a^2b\right)$$

$$= \frac{(s^2+4Rr+r^2)(4s)+2s(s^2-6Rr-3r^2)+\sum_{cyc}a^2b}{2s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \frac{51}{13} - \frac{3s^2+2Rr-r^2}{s^2+2Rr+r^2} \overset{?}{>} \frac{\sum_{cyc}a^2b}{2s(s^2+2Rr+r^2)}$$

$$\Leftrightarrow \frac{12s^2+76Rr+64r^2}{13(s^2+2Rr+r^2)} \boxed{\overset{?}{\underset{(*)}{>}}} \frac{\sum_{cyc}a^2b}{2s(s^2+2Rr+r^2)}$$

$$\text{Now, } \frac{\sum_{cyc}a^2b}{2s(s^2+2Rr+r^2)} \overset{CBS}{\le} \frac{\sqrt{(\sum_{cyc}a^2b^2)(\sum_{cyc}a^2)}}{2s(s^2+2Rr+r^2)} \overset{?}{<} \frac{12s^2+76Rr+64r^2}{13(s^2+2Rr+r^2)}$$

$$\Leftrightarrow 2s^2(12s^2+76Rr+64r^2)^2 \overset{?}{>} 169(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$\Leftrightarrow 119s^6+(5676Rr+2903r^2)s^4+r^2(3440R^2+18104Rr+8361r^2)s^2$$

$$+r^3(10816R^3+8112R^2r+2028Rr^2+169r^3) \overset{?}{>} 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10} \text{ and via AM} - \text{GM}, \sum_{cyc}\frac{a+b}{b+c} \ge 3 \text{ and so,}$$

$$3 \le \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} < 3+\frac{7+\sqrt{5}}{10}\ \forall\, \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equi$lateral$ (QED)