ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{w_a+w_b+w_c}{6}\ge\sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6)}{6(s^2+2Rr+r^2)^3}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{(b+c)^2}=\sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2}=2s.\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2}-\sum_{cyc}\frac{1}{b+c}$$

$$=\frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2-2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2}-\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}$$

$$=\frac{\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}\Rightarrow$$

$$\sum_{cyc}\frac{a}{(b+c)^2}\overset{(\bullet)}{=}\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)-(s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Now, }\sum_{cyc}w_a^2=\sum_{cyc}\frac{4bcs(s-a)}{(b+c)^2}=\sum_{cyc}\frac{bc((b+c)^2-a^2)}{(b+c)^2}=\sum_{cyc}\left(bc-\frac{a^2bc}{(b+c)^2}\right)$$

$$\overset{\text{via }(\bullet)}{=}s^2+4Rr+r^2$$

$$+2Rr.\frac{(s^2+2Rr+r^2)(5s^2+4Rr+r^2)+16s^2(s^2+2Rr+r^2)-(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$=\frac{s^6+3r^2s^4+r^2s^2(32R^2+40Rr+3r^2)+r^4(16R^2+8Rr+r^2)}{(s^2+2Rr+r^2)^2}$$

$$\therefore\frac{32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6}{(s^2+2Rr+r^2)^2}\boxed{\overset{(\bullet\bullet)}{=}}\sum_{cyc}w_a^2$$

$$\text{Again, }w_aw_bw_c=\prod_{cyc}\frac{2bc\cos\frac{A}{2}}{b+c}=\frac{8.16R^2r^2s^2.\frac{s}{4R}}{2s(s^2+2Rr+r^2)}=\frac{16Rr^2s^2}{s^2+2Rr+r^2}$$

$$\therefore\frac{s^2Rr^2}{s^2+2Rr+r^2}\boxed{\overset{(\bullet\bullet\bullet)}{=}}\frac{w_aw_bw_c}{16}\therefore(\bullet\bullet)\text{ and }(\bullet\bullet\bullet)\Rightarrow$$

$$\sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2+16R^2r^4+40Rs^2r^3+8Rr^5+s^6+3s^4r^2+3s^2r^4+r^6)}{6(s^2+2Rr+r^2)^3}}$$

$$=\sqrt[5]{\frac{w_aw_bw_c}{96}.\sum_{cyc}w_a^2}\le\frac{w_a+w_b+w_c}{6}\Leftrightarrow\frac{1}{6^5}\left(\sum_{cyc}x\right)^5\ge\frac{1}{96}.xyz\sum_{cyc}x^2$$

$$(x = w_a, y = w_b, z = w_c) \Leftrightarrow \left(\sum_{cyc} x\right)^5 \boxed{\overset{(*)}{\geq}} 81xyz\sum_{cyc} x^2$$

Assigning $y + z = M, z + x = N, x + y = P \Rightarrow M + N - P = 2z > 0, N + P - M = 2x > 0$ and $P + M - N = 2y > 0 \Rightarrow M + N > P, N + P > M, P + M > N \Rightarrow$ M, N, P form sides of a triangle with semiperimeter, circumradius and inradius $=$

$$s', R', r' \text{ (say) yielding } 2\sum_{cyc} x = \sum_{cyc} M = 2s' \Rightarrow \sum_{cyc} x = s' \to (1)$$

$$\Rightarrow x = s' - M, y = s' - N, z = s' - P \Rightarrow xyz = r'^2 s' \to (2) \text{ and}$$

via such substitutions, $\Rightarrow xyz = r'^2 s' \to (2)$ and via such substitutions,

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc}(s - M)(s - N)$$

$$\overset{\text{via }(1)}{=} s'^2 - 2(4R'r' + r'^2) \therefore \sum_{cyc} x^2 = s'^2 - 8R'r' - 2r'^2 \to (3)$$

$$\therefore (1), (2), (3) \Rightarrow (*) \Leftrightarrow s'^5 \geq 81r'^2 s'(s'^2 - 8R'r' - 2r'^2)$$

$$\Leftrightarrow s'^4 \boxed{\overset{(**)}{\geq}} 81r'^2(s'^2 - 8R'r' - 2r'^2)$$

Since $(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove :
$s^4 - 81r^2(s^2 - 8Rr - 2r^2) \geq 0$, it suffices to prove : LHS $\geq (s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (32R - 91r)s^2 \boxed{\overset{(***)}{\geq}} r(256R^2 - 808Rr - 137r^2)$$

$\boxed{Case\ 1}$ $32R - 91r \geq 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\geq} (32R - 91r)(16Rr - 5r^2)$

$$\overset{?}{\geq} r(256R^2 - 808Rr - 137r^2) \Leftrightarrow 32R^2 - 101Rr + 74r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(32R - 37r) \overset{?}{\geq} 0 \to \text{true} \because R \geq \frac{91r}{32} > 2r$$

$$\Rightarrow (***) \text{ is true (strict inequality)}$$

$\boxed{Case\ 2}$ $32R - 91r < 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(32R - 91r)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} r(256R^2 - 808Rr - 137r^2)$$

$$\Leftrightarrow 32t^3 - 123t^2 + 135t - 34 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(30t(t - 2) + 2t^2 + t + 17) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2r$$

$\Rightarrow (***)$ is true and combining both cases, $(***)$ is true $\forall \Delta ABC$

$$\therefore s^4 \geq 81r^2(s^2 - 8Rr - 2r^2) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \frac{w_a + w_b + w_c}{6}$$

$$\geq \sqrt[5]{\frac{s^2Rr^2(32R^2s^2r^2 + 16R^2r^4 + 40Rs^2r^3 + 8Rr^5 + s^6 + 3s^4r^2 + 3s^2r^4 + r^6)}{6(s^2 + 2Rr + r^2)^3}}$$

$\forall \Delta ABC$, " $=$ " iff ΔABC is equilateral (QED)