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In ΔABC the following relationship holds:

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\ge\frac{2\sqrt{3}}{\sqrt{s^2+r^2-8Rr}}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\sum w_a\le\sum\sqrt{s(s-a)}\overset{CBS}{\le}\sqrt{3s(s-a+s-b+s-c)}=s\sqrt{3}\ (1)$$

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\overset{CBS}{\ge}\frac{(1+1+1)^2}{\sum w_a}\overset{(1)}{\ge}\frac{9}{s\sqrt{3}}=\frac{3\sqrt{3}}{s}$$

$$We\ need\ to\ show:\ \frac{3\sqrt{3}}{s}\ge\frac{2\sqrt{3}}{\sqrt{s^2+r^2-8Rr}}$$

$$3\sqrt{s^2+r^2-8Rr}\ge 2s$$

$$9s^2+9r^2-72Rr\ge 4s^2$$

$$5s^2-72Rr+9r^2\ge 0$$

$$5(16Rr-5r^2)-72Rr+9r^2\overset{Gerretsen}{\ge}0$$

$$8Rr\ge 16r^2,\ R\ge 2r\ true\ Euler$$

$$Equality\ holds\ for\ an\ equilateral\ triangle.$$