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In $\triangle ABC$ the following relationship holds:

$$\left(\sum \csc A\right) \left(\sum \cot \frac{A}{2}\right) \geq 2 \sum \cot \frac{A}{2} \cot \frac{B}{2}$$

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$$\begin{aligned} \left(\sum \csc A\right) &= \sum \frac{1}{\sin A} \stackrel{\text{Bergstrom}}{\geq} \frac{(1+1+1)^2}{\sum \sin A} = \frac{9}{\frac{s}{R}} = \frac{9R}{s} \quad (1) \\ \left(\sum \cot \frac{A}{2}\right) &= \sum \frac{s}{r_a} = \frac{s}{r} \quad (2) \end{aligned}$$

$$\begin{aligned} 2 \sum \cot \frac{A}{2} \cot \frac{B}{2} &= 2 \sum \frac{s}{r_a} \cdot \frac{s}{r_b} = 2s^2 \frac{\sum r_a}{r_a r_b r_c} = \\ &= 2s^2 \frac{4R+r}{s^2 r} = \frac{8R+2r}{r} \stackrel{\text{Euler}}{\leq} \frac{(8R+R)}{r} = \frac{9R}{r} \quad (A) \end{aligned}$$

$$\left(\sum \csc A\right) \left(\sum \cot \frac{A}{2}\right) \stackrel{(1) \& (2)}{\geq} \frac{9R}{s} \cdot \frac{s}{r} = \frac{9R}{r} \quad (B)$$

From (A) and (B) we get $\left(\sum \csc A\right) \left(\sum \cot \frac{A}{2}\right) \geq 2 \sum \cot \frac{A}{2} \cot \frac{B}{2}$

Equality holds for $A = B = C$