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In Δ ABC with $n_a, n_b, n_c \to$ Nagel cevians, g_a, g_b, g_c $\to$ Gergonne cevians, holds :

$$\left(\sqrt{3}-1\right)(n_a+n_b+n_c)+\left(2-\sqrt{3}\right)(g_a+g_b+g_c)\geq\sqrt{3}s$$

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$$\left(\sqrt{3}-1\right)(n_a+n_b+n_c)+\left(2-\sqrt{3}\right)(g_a+g_b+g_c)\geq\sqrt{3}s$$

$$\Leftrightarrow\left(\sqrt{3}-1\right)\left(\sum_{cyc}n_a\right)+\frac{\left(\sqrt{3}-1\right)^2}{2}\left(\sum_{cyc}g_a\right)\geq\sqrt{3}s$$

$$\Leftrightarrow\frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}-1\right)^2}.\left(\sum_{cyc}n_a\right)+\frac{1}{2}\left(\sum_{cyc}g_a\right)\geq\frac{\sqrt{3}s}{2\left(2-\sqrt{3}\right)}$$

$$\Leftrightarrow\left(\sqrt{3}+1\right)\left(\sum_{cyc}n_a\right)+\sum_{cyc}g_a\boxed{\overset{(*)}{\geq}}2\sqrt{3}s+3s\text{ and }\because\sum_{cyc}n_a+\sum_{cyc}g_a$$

$$\overset{\text{Bogdan Fustei (2019)}}{\geq}2\sum_{cyc}m_a\ \therefore\text{ in order to prove }(*),\text{ it suffices to prove :}$$

$$\sqrt{3}.\left(\sum_{cyc}n_a\right)+2\sum_{cyc}m_a\overset{?}{\geq}2\sqrt{3}s+3s\overset{\text{squaring}}{\Leftrightarrow}$$

$$3\left(\sum_{cyc}n_a\right)^2+4\left(\sum_{cyc}m_a\right)^2+4\sqrt{3}.\left(\sum_{cyc}n_a\right)\left(\sum_{cyc}m_a\right)\boxed{\underset{(**)}{\overset{?}{\geq}}}21s^2+12\sqrt{3}s^2$$

$$\text{Now, }3\left(\sum_{cyc}n_a\right)^2+4\left(\sum_{cyc}m_a\right)^2-21s^2\geq3(9s^2-80Rr-2r^2)+$$

$$4(4s^2-28Rr+29r^2)-21s^2=22(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$$\left(\begin{array}{c}\because\left(\displaystyle\sum_{cyc}n_a\right)^2\geq9s^2-80Rr-2r^2;\text{ reference : Inequality in Triangle by Mohamed Amine}\\ \text{Ben Ajiba}-74,\text{ published at www.ssmrmh.ro and}\\ \because\left(\displaystyle\sum_{cyc}m_a\right)^2\overset{\text{Chu and Yang}}{\geq}4s^2-28Rr+29r^2\end{array}\right)$$

$$\therefore 3\left(\sum_{cyc} n_a\right)^2 + 4\left(\sum_{cyc} m_a\right)^2 \overset{①}{\geq} 21s^2 \ \& \ also, \left(\sum_{cyc} n_a\right)^2 \cdot \left(\sum_{cyc} m_a\right)^2 - 9s^4$$

$$\overset{\substack{\text{Ben Ajiba} \\ \text{and} \\ \text{Chu and Yang}}}{\geq} (9s^2 - 80Rr - 2r^2)(4s^2 - 28Rr + 29r^2) - 9s^4 \text{ and } \because P =$$

$$27(s^2 - 16Rr + 5r^2)^2 + r(292R - 17r)(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove : } \left(\sum_{cyc} n_a\right)^2 \cdot \left(\sum_{cyc} m_a\right)^2 - 9s^4 \overset{?}{\geq} 0, \text{it suffices to prove :}$$

$$(9s^2 - 80Rr - 2r^2)(4s^2 - 28Rr + 29r^2) - 9s^4 \overset{?}{\geq} P \Leftrightarrow 324r^3(R - 2r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true via Euler} \therefore \left(\sum_{cyc} n_a\right)^2 \cdot \left(\sum_{cyc} m_a\right)^2 \geq 9s^4$$

$$\Rightarrow 4\sqrt{3}.\left(\sum_{cyc} n_a\right)\left(\sum_{cyc} m_a\right) \overset{②}{\geq} 12\sqrt{3}s^2 \therefore ① + ② \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\sqrt{3} - 1\right)(n_a + n_b + n_c) + \left(2 - \sqrt{3}\right)(g_a + g_b + g_c) \geq \sqrt{3}s \ \forall \ \Delta \ ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$