ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c → Nagel cevians, the following relationship holds :

$$\sum_{cyc}\left(n_a(n_b+n_c-n_a)\right)\ge\sum_{cyc}h_a^2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$n_b^2+n_c^2+2n_bn_c-n_a^2\overset{\text{Bogdan Fustei}}{\ge}s(s-b)+\frac{s}{b}(c-a)^2+s(s-c)$$
$$+\frac{s}{c}(a-b)^2+2g_bg_c-s(s-a)-\frac{s}{a}(b-c)^2\ge s(s-b)+\frac{s}{b}(c-a)^2$$
$$+s(s-c)+\frac{s}{c}(a-b)^2+2(s-b)(s-c)-s(s-a)-\frac{s}{a}(b-c)^2$$
$$\left(\because g_a^2\overset{\text{Bogdan Fustei}}{=}(s-a)^2+2rh_a\Rightarrow g_a>s-a\text{ and analogs}\right)$$
$$=\frac{\left(\begin{array}{c}\left(\sum_{cyc}x\right)(y+z-x).\prod_{cyc}(y+z)+\\ \left(\sum_{cyc}x\right)\left((z-x)^2(x+y)(y+z)+(x-y)^2(y+z)(z+x)-(y-z)^2(z+x)(x+y)\right)+\\ 2yz.\prod_{cyc}(y+z)\end{array}\right)}{\prod_{cyc}(y+z)}$$
$$(x=s-a,y=s-b,z=s-c\text{ and hence : }a=y+z,b=z+x,c=x+y)$$
$$=\frac{\begin{array}{c}xy\left(x^3+y^3-xy(x+y)\right)+zx\left(z^3+x^3-zx(z+x)\right)+2x^3yz+4xy^3z+10xy^2z^2+\\ 4xyz^3+y^4z+yz^4+9y^3z^2+9y^2z^3\end{array}}{\prod_{cyc}(y+z)}$$
$$>0\ (\because x^3+y^3\ge xy(x+y)\text{ etc})\Rightarrow n_b+n_c>n_a\text{ and similarly, }n_c+n_a>n_b$$
$$\text{and }n_a+n_b>n_c\Rightarrow n_a,n_b,n_c\text{ form sides of a triangle }\therefore\sum_{cyc}\left(n_a(n_b+n_c-n_a)\right)$$
$$\overset{\text{Hadwiger-Finsler}}{\ge}\sqrt{3}.\sqrt{2\sum_{cyc}n_b^2n_c^2-\sum_{cyc}n_a^4}$$
$$=\sqrt{3}.\left(4\sum_{cyc}n_b^2n_c^2-\left(\sum_{cyc}n_a^2\right)^2\right)\overset{\text{Bogdan Fustei}}{=}$$
$$\sqrt{3}.\left(4s^4\sum_{cyc}\left(\left(1-\frac{r}{R}\sec^2\frac{B}{2}\right)\left(1-\frac{r}{R}\sec^2\frac{C}{2}\right)\right)-s^4\left(\sum_{cyc}\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)\right)^2\right)$$

$$= \sqrt{3}.\sqrt{4s^4\left(3 - \frac{2r}{R}.\frac{s^2+(4R+r)^2}{s^2} + \frac{r^2}{R^2}.\frac{16R^2}{s^2}.\frac{4R+r}{2R}\right) - s^4\left(3 - \frac{r}{R}.\frac{s^2+(4R+r)^2}{s^2}\right)^2}$$

$$\stackrel{?}{\geq} \sum_{cyc} h_a^2 = \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{4R^2} \stackrel{squaring}{\Leftrightarrow}$$

$$192s^2R^3\left((3R-2r)s^2 - 2r(4R+r)^2 + 8r^2(4R+r)\right)$$

$$-((s^2+4Rr+r^2)^2 - 16Rrs^2)^2 - 48R^2\left((3R-r)s^2 - r(4R+r)^2\right)^2 \boxed{\stackrel{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because$$

$$P = -s^4(s^4 - (4R^2+20Rr-2r^2)s^2 + r(4R+r)^3)$$
$$-(4R^2+4Rr+2r^2)s^2(s^4 - (4R^2+20Rr-2r^2)s^2 + r(4R+r)^3) +$$
$$(128R^4 - 128R^3r - 176R^2r^2 - 4Rr^3 - r^4)(s^2-16Rr+5r^2)^2 +$$
$$4r(704R^5 - 272R^4r - 764R^3r^2 + 437R^2r^3 + 5Rr^4 + 2r^5)(s^2-16Rr+5r^2)$$

$$\stackrel{\substack{\text{Double Rouche} \\ \text{and} \\ \text{Gerretsen}}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \stackrel{?}{\geq} P$$

$$\Leftrightarrow 2368t^5 - 6736t^4 + 4572t^3 - 1137t^2 - 12t - 4 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)(1368t^4 + 1000t^2(t-2) + 572t^2 + 7t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true } \therefore \sum_{cyc}\left(n_a(n_b+n_c-n_a)\right) \geq \sum_{cyc} h_a^2 \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$