ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$s(n_a+n_b+n_c)\ge(2-\sqrt{3})(n_a^2+n_b^2+n_c^2)+2(\sqrt{3}-1)s^2$$

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$$s(n_a+n_b+n_c)\ge(2-\sqrt{3})(n_a^2+n_b^2+n_c^2)+2(\sqrt{3}-1)s^2 \overset{\text{squaring}}{\Leftrightarrow}$$

$$s^2\left(\sum_{cyc}n_a\right)^2\ge\frac{(\sqrt{3}-1)^4}{4}\left(\sum_{cyc}n_a^2\right)^2+4(\sqrt{3}-1)^2s^4+2(\sqrt{3}-1)^3s^2\left(\sum_{cyc}n_a^2\right)$$

$$\Leftrightarrow s^2\left(\sum_{cyc}n_a\right)^2\left(\frac{2+\sqrt{3}}{2}\right)\ge\left(\frac{2-\sqrt{3}}{2}\right)\left(\sum_{cyc}n_a^2\right)^2+4s^4+(2\sqrt{3}-2)s^2\left(\sum_{cyc}n_a^2\right)$$

$$\Leftrightarrow\sqrt{3}\left(s^2\left(\sum_{cyc}n_a\right)^2+\left(\sum_{cyc}n_a^2\right)^2-4s^2\left(\sum_{cyc}n_a^2\right)\right)+$$

$$2s^2\left(\sum_{cyc}n_a\right)^2-2\left(\sum_{cyc}n_a^2\right)^2-8s^4+4s^2\left(\sum_{cyc}n_a^2\right)\boxed{\overset{(*)}{\ge}}0$$

$\boxed{\text{Case 1}}$ $s^2\left(\sum_{cyc}n_a\right)^2+\left(\sum_{cyc}n_a^2\right)^2-4s^2\left(\sum_{cyc}n_a^2\right)\ge0\ \&\ \because\sqrt{3}>\frac{5}{3}\ \therefore$ LHS of $(*)\ge$

$$\frac{5\left(s^2(\sum_{cyc}n_a)^2+(\sum_{cyc}n_a^2)^2-4s^2(\sum_{cyc}n_a^2)\right)}{3}+2s^2\left(\sum_{cyc}n_a\right)^2-2\left(\sum_{cyc}n_a^2\right)^2$$

$$-8s^4+4s^2\left(\sum_{cyc}n_a^2\right)\overset{?}{\ge}0$$

$$\Leftrightarrow 11s^2\left(\sum_{cyc}n_a\right)^2-\left(\sum_{cyc}n_a^2\right)^2-8s^2\left(\sum_{cyc}n_a^2\right)-24s^4\boxed{\underset{(**)}{\overset{?}{\ge}}}0$$

Now, LHS of $(**)\overset{\text{Ben Ajiba}}{\ge}11s^2(9s^2-80Rr-2r^2)-\frac{\left((3R-r)s^2-r(4R+r)^2\right)^2}{R^2}$

$$-8s^2\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)-24s^4\overset{?}{\ge}0\Leftrightarrow 11s^2R^2(9s^2-80Rr-2r^2)-$$

$$\left((3R-r)s^2-r(4R+r)^2\right)^2-8s^2R\left((3R-r)s^2-r(4R+r)^2\right)-24s^4R^2\boxed{\underset{(1)}{\overset{?}{\ge}}}0\ \&\ \because P$$

$= (42R^2 + 14Rr - r^2)(s^2 - 16Rr + 5r^2)^2 + 2r(344R^3 + 43R^2r - 87Rr^2 + 4r^3) *$
$(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\geq} 0 \therefore$ **in order to prove** (1), **it suffices to prove :**
LHS of $(1) \overset{?}{\geq} P \Leftrightarrow 8r^3(R - 2r)(102R^2 - 29Rr + r^2) \overset{?}{\geq} 0 \to$ **true** $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow$

$(1) \Rightarrow (**) \Rightarrow (*)$ **is true** $\boxed{\textit{Case 2}}$ $s^2\left(\sum_{cyc} n_a\right)^2 + \left(\sum_{cyc} n_a^2\right)^2 - 4s^2\left(\sum_{cyc} n_a^2\right) \leq 0$

and $\because \sqrt{3} < \frac{7}{4} \therefore$ **LHS of** $(*) \geq \frac{7\left(s^2\left(\sum_{cyc} n_a\right)^2 + \left(\sum_{cyc} n_a^2\right)^2 - 4s^2\left(\sum_{cyc} n_a^2\right)\right)}{4} +$

$2s^2\left(\sum_{cyc} n_a\right)^2 - 2\left(\sum_{cyc} n_a^2\right)^2 - 8s^4 + 4s^2\left(\sum_{cyc} n_a^2\right) \overset{?}{\geq} 0$

$\Leftrightarrow 15s^2\left(\sum_{cyc} n_a\right)^2 - \left(\sum_{cyc} n_a^2\right)^2 - 12s^2\left(\sum_{cyc} n_a^2\right) - 32s^4 \boxed{\overset{?}{\underset{(***)}{\geq}}} 0 \ \&\ (**) \Rightarrow$

to prove $(***)$, **it suffices to prove : LHS of** $(***) \overset{?}{\geq}$ **LHS of** $(**) \Leftrightarrow s^2\left(\sum_{cyc} n_a\right)^2$

$\overset{?}{\geq} s^2\left(\sum_{cyc} n_a^2\right) + 2s^4$ *and* $\because s^2\left(\sum_{cyc} n_a\right)^2 \overset{\text{Ben Ajiba}}{\geq} s^2(9s^2 - 80Rr - 2r^2) \therefore$

suffices to prove : $Rs^2(9s^2 - 80Rr - 2r^2) - s^2\left((3R - r)s^2 - r(4R + r)^2\right) - 2Rs^4$
$\boxed{\overset{?}{\underset{(2)}{\geq}}} 0$ *and* $\because Q = s^2(4R + r)(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\geq} 0 \therefore$ **in order to prove** (2),

it suffices to prove : LHS of $(2) \overset{?}{\geq} Q \Leftrightarrow 2r^2s^2(R - 2r) \overset{?}{\geq} 0 \to$ **true** $\because R \overset{\text{Euler}}{\geq} 2r$
$\Rightarrow (2) \Rightarrow (***) \Rightarrow (*)$ **is true** $\therefore$ **combining both** ***cases***, $(*)$ **is true** $\forall \Delta ABC \therefore$
$s(n_a + n_b + n_c) \geq (2 - \sqrt{3})(n_a^2 + n_b^2 + n_c^2) + 2(\sqrt{3} - 1)s^2 \ \forall \Delta ABC,$
$'' ='' $ **iff** ΔABC **is equi*lateral*** (QED)