ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$ with n_a, n_b, n_c

$\rightarrow$ Nagel cevians, the following relationship holds :

$$n_a n_b + n_b n_c + n_c n_a \geq 3s^2 - 32Rr + 10r^2$$

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$$n_b^2 + n_c^2 + 2n_b n_c - n_a^2 \overset{\text{Bogdan Fustei}}{\geq} s(s-b) + \frac{s}{b}(c-a)^2 + s(s-c)$$

$$+\frac{s}{c}(a-b)^2 + 2g_b g_c - s(s-a) - \frac{s}{a}(b-c)^2 \geq s(s-b) + \frac{s}{b}(c-a)^2$$

$$+s(s-c) + \frac{s}{c}(a-b)^2 + 2(s-b)(s-c) - s(s-a) - \frac{s}{a}(b-c)^2$$

$$\left(\because g_a^2 \overset{\text{Bogdan Fustei}}{=} (s-a)^2 + 2rh_a \Rightarrow g_a > s-a \text{ and analogs}\right)$$

$$= \frac{\left(\begin{gathered}\left(\sum_{cyc} x\right)(y+z-x).\prod_{cyc}(y+z) + \\ \left(\sum_{cyc} x\right)\left((z-x)^2(x+y)(y+z) + (x-y)^2(y+z)(z+x) - (y-z)^2(z+x)(x+y)\right) + \\ 2yz.\prod_{cyc}(y+z)\end{gathered}\right)}{\prod_{cyc}(y+z)}$$

$$(x = s-a, y = s-b, z = s-c \text{ and hence} : a = y+z, b = z+x, c = x+y)$$

$$= \frac{\begin{gathered}xy\left(x^3 + y^3 - xy(x+y)\right) + zx\left(z^3 + x^3 - zx(z+x)\right) + 2x^3yz + 4xy^3z + 10xy^2z^2 + \\ 4xyz^3 + y^4z + yz^4 + 9y^3z^2 + 9y^2z^3\end{gathered}}{\prod_{cyc}(y+z)}$$

$$> 0 \; (\because x^3 + y^3 \geq xy(x+y) \text{ etc}) \Rightarrow n_b + n_c > n_a \text{ and similarly}, n_c + n_a > n_b$$

$$\text{and } n_a + n_b > n_c \Rightarrow n_a, n_b, n_c \text{ form sides of a triangle} \therefore \sum_{cyc}\left(n_a(n_b + n_c - n_a)\right)$$

$$\overset{\text{Hadwiger-Finsler}}{\geq} \sqrt{3}.\sqrt{2\sum_{cyc} n_b^2 n_c^2 - \sum_{cyc} n_a^4}$$

$$= \sqrt{3}.\left(4\sum_{cyc} n_b^2 n_c^2 - \left(\sum_{cyc} n_a^2\right)^2\right) \overset{\text{Bogdan Fustei}}{=}$$

$$\sqrt{3}.\left(4s^4 \sum_{cyc}\left(\left(1 - \frac{r}{R}\sec^2\frac{B}{2}\right)\left(1 - \frac{r}{R}\sec^2\frac{C}{2}\right)\right) - s^4\left(\sum_{cyc}\left(1 - \frac{r}{R}\sec^2\frac{A}{2}\right)\right)^2\right)$$

$$= \sqrt{3}.\sqrt{4s^4\left(3 - \frac{2r}{R}.\frac{s^2 + (4R+r)^2}{s^2} + \frac{r^2}{R^2}.\frac{16R^2}{s^2}.\frac{4R+r}{2R}\right) - s^4\left(3 - \frac{r}{R}.\frac{s^2 + (4R+r)^2}{s^2}\right)^2}$$

$$\overset{?}{\geq} 6s^2 - 64Rr + 20r^2 - \sum_{cyc} n_a^2 = \frac{(3R+r)s^2 - r(48R^2 - 28Rr - r^2)}{R}$$

$$\Leftrightarrow 12s^2 . \frac{(3R-2r)s^2 - 2r(4R+r)^2 + 8r^2(4R+r)}{R} - 3 . \frac{\left((3R-r)s^2 - r(4R+r)^2\right)^2}{R^2}$$

$$\overset{?}{\geq} \frac{\left((3R+r)s^2 - r(48R^2 - 28Rr - r^2)\right)^2}{R^2}$$

$$\left(\begin{array}{c} \because (3R+r)s^2 - r(48R^2 - 28Rr - r^2) \overset{\text{Gerretsen}}{\geq} (3R+r)(16Rr - 5r^2) \\ -r(48R^2 - 28Rr - r^2) = r^2(29R - 4r) > 0 \end{array} \right) \Leftrightarrow$$

$$12s^2R\left((3R-2r)s^2 - 2r(4R+r)^2 + 8r^2(4R+r)\right) - 3\left((3R-r)s^2 - r(4R+r)^2\right)^2$$

$$- \left((3R+r)s^2 - r(48R^2 - 28Rr - r^2)\right)^2 \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because P =$$

$$-4r(3R+r)(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3) +$$

$$4Rr(36R^2 - 22Rr - 19r^2)(s^2 - 16Rr + 5r^2) \overset{\substack{\text{Double Rouche} \\ \text{and} \\ \text{Gerretsen}}}{\geq} 0 \therefore \text{in order to}$$

$$\text{prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P \Leftrightarrow 24Rr^3(R-2r)(26R-7r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true} \therefore n_a n_b + n_b n_c + n_c n_a \geq 3s^2 - 32Rr + 10r^2$$

$$\forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$