ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c

$\rightarrow$ Nagel cevians, the following relationship holds :

$$n_a + n_b + n_c - 9r \geq 3 \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a+b+c}$$

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$$n_a + n_b + n_c - 9r \geq 3 \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a+b+c} \Leftrightarrow$$

$$\left(\sum_{\text{cyc}} n_a\right)^2 \geq 81r^2 + \frac{9(s^2 - 12Rr - 3r^2)^2}{s^2} + \frac{54r(s^2 - 12Rr - 3r^2)}{s}$$

$$\Leftrightarrow s^2\left(\sum_{\text{cyc}} n_a\right)^2 - 9(s^2 - 12Rr - 3r^2)^2 - 81r^2s^2 - 54rs(s^2 - 12Rr - 3r^2) \geq 0$$

$$\text{and} \because s^2\left(\sum_{\text{cyc}} n_a\right)^2 \overset{\text{Ben Ajiba}}{\geq} s^2(9s^2 - 80Rr - 2r^2) \therefore \text{it sufices to prove :}$$

$$s^2(9s^2 - 80Rr - 2r^2) - 9(s^2 - 12Rr - 3r^2)^2 - 81r^2s^2 - 54rs(s^2 - 12Rr - 3r^2)$$

$$\overset{?}{\geq} 0 \Leftrightarrow (136R - 29r)s^2 - 81r(4R + r)^2 \overset{?}{\geq} 54s(s^2 - 12Rr - 3r^2)$$

$$\Leftrightarrow \left((136R - 29r)s^2 - 81r(4R + r)^2\right)^2 \boxed{\overset{?}{\geq}_{(*)}} 2916s^2(s^2 - 12Rr - 3r^2)^2$$

$$\left(\begin{gathered}\because (136R - 29r)s^2 - 81r(4R + r)^2 \overset{\text{Gerretsen}}{\geq} (136R - 29r)(16Rr - 5r^2) - 81r(4R + r)^2 \\ = 16r(R - 2r)(55R - 2r) \overset{\text{Euler}}{\geq} 0\end{gathered}\right)$$

$$\text{and} \because P = -2916s^2(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) +$$

$$(6832R^2 + 3776Rr + 24169r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\substack{\text{Double-Rouche} \\ \text{and} \\ \text{Gerretsen}}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } (*), \text{it suffices to prove : LHS of } (*) - \text{RHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (3296R^3 - 20532R^2r + 36015Rr^2 - 16270r^3)s^2 \boxed{\overset{?}{\geq}_{(**)}}$$

$$r(4336R^4 - 112880R^3r + 320253R^2r^2 - 242351Rr^3 + 37354r^4)$$

$\boxed{\text{Case 1}}$ $3296R^3 - 20532R^2r + 36015Rr^2 - 16270r^3 \geq 0$ *and* then, LHS of $(**)$

$$\overset{\text{Gerretsen}}{\geq} (3296R^3 - 20532R^2r + 36015Rr^2 - 16270r^3)(16Rr - 5r^2) \overset{?}{\geq}$$

$$\text{RHS of } (**) \Leftrightarrow 48400t^4 - 232112t^3 + 358647t^2 - 198044t + 43996 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)^2(29144t^2 + 19256t(t-2) + 10999) \overset{?}{\geq} 0$$

$$\to \mathbf{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true}$$

$\boxed{Case\ 2}$ $3296R^3 - 20532R^2r + 36015Rr^2 - 16270r^3 < 0$ *and* then, LHS of $(**)$

$$\overset{\text{Gerretsen}}{\geq} (3296R^3 - 20532R^2r + 36015Rr^2 - 16270r^3)(4R^2 + 4R + 3r^2)$$

$$\overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 13184t^5 - 73280t^4 + 184700t^3 - 302869t^2 + 285316t - 86164 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)^2(2912t^3 + 10272t^2(t-2) + 49788 - 21541) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (**)$ is true*and* combining both *c*ases, $(**) \Rightarrow (*)$ is true $\forall\, \Delta\, ABC$

$$\therefore n_a + n_b + n_c - 9r \geq 3.\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a+b+c}\ \forall\, \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equi*l*ater*al* (QED)