ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c

$\to$ Nagel cevians, the following relationship holds :

$$(n_a+n_b+n_c)^2 \ge 9s^2-80Rr-2r^2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$n_b^2+n_c^2+2n_bn_c-n_a^2 \overset{\text{Bogdan Fustei}}{\ge} s(s-b)+\frac{s}{b}(c-a)^2+s(s-c)$$
$$+\frac{s}{c}(a-b)^2+2g_bg_c-s(s-a)-\frac{s}{a}(b-c)^2 \ge s(s-b)+\frac{s}{b}(c-a)^2$$
$$+s(s-c)+\frac{s}{c}(a-b)^2+2(s-b)(s-c)-s(s-a)-\frac{s}{a}(b-c)^2$$
$$\left(\because g_a^2 \overset{\text{Bogdan Fustei}}{=} (s-a)^2+2rh_a \Rightarrow g_a > s-a \text{ and analogs}\right)$$

$$=\frac{\left(\begin{gathered}\left(\sum_{cyc}x\right)(y+z-x).\prod_{cyc}(y+z)+\\ \left(\sum_{cyc}x\right)\left((z-x)^2(x+y)(y+z)+(x-y)^2(y+z)(z+x)-(y-z)^2(z+x)(x+y)\right)+\\ 2yz.\prod_{cyc}(y+z)\end{gathered}\right)}{\prod_{cyc}(y+z)}$$

$$(x=s-a, y=s-b, z=s-c \text{ and hence : } a=y+z, b=z+x, c=x+y)$$

$$=\frac{\begin{gathered}xy\left(x^3+y^3-xy(x+y)\right)+zx\left(z^3+x^3-zx(z+x)\right)+2x^3yz+4xy^3z+10xy^2z^2+\\ 4xyz^3+y^4z+yz^4+9y^3z^2+9y^2z^3\end{gathered}}{\prod_{cyc}(y+z)}$$

> 0 $(\because x^3+y^3 \ge xy(x+y)$ etc$) \Rightarrow n_b+n_c > n_a$ and similarly, $n_c+n_a > n_b$ and $n_a+n_b > n_c \Rightarrow n_a, n_b, n_c$ form sides of a triangle $\therefore \sum_{cyc}\left(n_a(n_b+n_c-n_a)\right)$

$$\overset{\text{Hadwiger-Finsler}}{\ge} \sqrt{3}.\sqrt{2\sum_{cyc}n_b^2n_c^2-\sum_{cyc}n_a^4}$$

$$=\sqrt{3}.\sqrt{4\sum_{cyc}n_b^2n_c^2-\left(\sum_{cyc}n_a^2\right)^2} \overset{\text{Bogdan Fustei}}{=}$$

$$\sqrt{3}.\sqrt{4s^4\sum_{cyc}\left(\left(1-\frac{r}{R}\sec^2\frac{B}{2}\right)\left(1-\frac{r}{R}\sec^2\frac{C}{2}\right)\right)-s^4\left(\sum_{cyc}\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)\right)^2}$$

$$=\sqrt{3}.\sqrt{4s^4\left(3-\frac{2r}{R}.\frac{s^2+(4R+r)^2}{s^2}+\frac{r^2}{R^2}.\frac{16R^2}{s^2}.\frac{4R+r}{2R}\right)-s^4\left(3-\frac{r}{R}.\frac{s^2+(4R+r)^2}{s^2}\right)^2}$$

$$\overset{?}{\ge} 9s^2-80Rr-2r^2-2\sum_{cyc}n_a^2=\frac{(3R+2r)s^2-r(48R^2-14Rr-2r^2)}{R}$$

$$\Leftrightarrow 12s^2.\frac{(3R-2r)s^2-2r(4R+r)^2+8r^2(4R+r)}{R}-3.\frac{\left((3R-r)s^2-r(4R+r)^2\right)^2}{R^2}$$

$$\overset{?}{\geq}\frac{\left((3R+2r)s^2-r(48R^2-14Rr-2r^2)\right)^2}{R^2}$$

$$\left(\begin{matrix}\because (3R+2r)s^2-r(48R^2-14Rr-2r^2)\overset{\text{Gerretsen}}{\geq}(3R+2r)(16Rr-5r^2)\\ -r(48R^2-14Rr-2r^2)=r^2(31R-8r)>0\end{matrix}\right)\Leftrightarrow$$

$$12s^2R\left((3R-2r)s^2-2r(4R+r)^2+8r^2(4R+r)\right)-3\left((3R-r)s^2-r(4R+r)^2\right)^2$$

$$-\left((3R+2r)s^2-r(48R^2-14Rr-2r^2)\right)^2\boxed{\underset{(*)}{\overset{?}{\geq}}}0 \text{ and } \because P=$$

$$-r(18R+7r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)+$$

$$2Rr(60R^2-20Rr-65r^2)(s^2-16Rr+5r^2)\overset{\substack{\text{Double Rouche}\\ \text{and}\\ \text{Gerretsen}}}{\geq}0\ \therefore \text{ in order to}$$

$$\text{prove } (*), \text{it suffices to prove}: \text{LHS of } (*)\overset{?}{\geq}P\Leftrightarrow 324Rr^3(R-2r)(2R-r)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow (*) \text{ is true} \therefore (n_a+n_b+n_c)^2\geq 9s^2-80Rr-2r^2$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$