

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC with n_a, n_b, n_c

→ Nagel cevians, following relationship holds :

$$n_a + n_b + n_c + \frac{2\sqrt{3} - 3}{3} (h_a + h_b + h_c) \geq 2s$$

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$$n_a + n_b + n_c \stackrel{?}{\geq} 2s - 3(2\sqrt{3} - 3)r \Leftrightarrow \sum_{cyc} n_a + 6\sqrt{3}r \stackrel{?}{\geq} 2s + 9r \quad \text{squaring} \Leftrightarrow$$

$$\left(\sum_{cyc} n_a \right)^2 + 108r^2 + 12\sqrt{3}r \left(\sum_{cyc} n_a \right) \stackrel{?}{\geq} 4s^2 + 81r^2 + 36rs \quad (*)$$

$$\text{Now, via Ben Ajiba, } \left(\sum_{cyc} n_a \right)^2 \geq 9s^2 - 80Rr - 2r^2 = 3s^2 + 6(s^2 - 16Rr + 5r^2) +$$

$$16r(R - 2r) \stackrel{\text{Gerretsen and Euler}}{\geq} 3s^2 \Rightarrow 12\sqrt{3}r \left(\sum_{cyc} n_a \right) \geq 36rs \text{ \& so, LHS of } (*) - \text{RHS of } (*)$$

$$\geq 9s^2 - 80Rr - 2r^2 + 108r^2 + 36rs - 4s^2 - 81r^2 - 36rs = 5(s^2 - 16Rr + 5r^2)$$

$$\stackrel{\text{Gerretsen}}{\geq} 0 \Rightarrow (*) \text{ is true } \therefore n_a + n_b + n_c - 2s \geq -3(2\sqrt{3} - 3)r \stackrel{?}{\geq}$$

$$-\frac{2\sqrt{3} - 3}{3} (h_a + h_b + h_c) \Leftrightarrow h_a + h_b + h_c \stackrel{?}{\geq} 9r \rightarrow \text{true}$$

$$\therefore n_a + n_b + n_c + \frac{2\sqrt{3} - 3}{3} (h_a + h_b + h_c) \geq 2s \quad \forall \Delta ABC,$$

"=" iff ΔABC is equilateral (QED)