ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c

$\rightarrow$ Nagel cevians, the following relationship holds :

$$\left(2\sqrt{3}-3\right)(n_a+n_b+n_c)+\left(4-2\sqrt{3}\right)(m_a+m_b+m_c)\geq\sqrt{3}s$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\left(2\sqrt{3}-3\right)(n_a+n_b+n_c)+\left(4-2\sqrt{3}\right)(m_a+m_b+m_c)\geq\sqrt{3}s\Leftrightarrow$$
$$\left(\frac{2\sqrt{3}-3}{2-\sqrt{3}}\right)(n_a+n_b+n_c)+\left(\frac{4-2\sqrt{3}}{2-\sqrt{3}}\right)(m_a+m_b+m_c)\geq\frac{\sqrt{3}s}{2-\sqrt{3}}$$
$$\Leftrightarrow\left(\sqrt{3}\sum_{cyc}n_a+2\sum_{cyc}m_a\right)^2\geq\left(2\sqrt{3}s+3s\right)^2\Leftrightarrow$$
$$3\left(\sum_{cyc}n_a\right)^2+4\left(\sum_{cyc}m_a\right)^2+4\sqrt{3}\left(\sum_{cyc}n_a\right)\left(\sum_{cyc}m_a\right)\boxed{\overset{(*)}{\geq}}21s^2+12\sqrt{3}s^2$$

$$\text{Now, }\left(\sum_{cyc}n_a\right)\left(\sum_{cyc}m_a\right)\overset{\substack{\text{Ben Ajiba}\\ \text{and}\\ \text{Chu-Yang}}}{\geq}\sqrt{(9s^2-80Rr-2r^2)(4s^2-28Rr+29r^2)}\overset{?}{\geq}3s^2$$

$$\Leftrightarrow(9s^2-80Rr-2r^2)(4s^2-28Rr+29r^2)-9s^4\boxed{\overset{?}{\underset{(\bullet)}{\geq}}}0\text{ and }\because P=$$

$$27(s^2-16Rr+5r^2)^2+(292Rr-17r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order}$$

to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet)\overset{?}{\geq}P\Leftrightarrow 324r^3(R-2r)\overset{?}{\geq}0\rightarrow$ true

$$\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet)\text{ is true}\therefore 4\sqrt{3}\left(\sum_{cyc}n_a\right)\left(\sum_{cyc}m_a\right)\geq 12\sqrt{3}s^2\rightarrow(1)\text{ and again,}$$

$$3\left(\sum_{cyc}n_a\right)^2+4\left(\sum_{cyc}m_a\right)^2\overset{\substack{\text{Ben Ajiba}\\ \text{and}\\ \text{Chu-Yang}}}{\geq}3(9s^2-80Rr-2r^2)$$

$$+4(4s^2-28Rr+29r^2)\overset{?}{\geq}21s^2\Leftrightarrow 22(s^2-16Rr+5r^2)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\text{via Gerretsen}\therefore 3\left(\sum_{cyc}n_a\right)^2+4\left(\sum_{cyc}m_a\right)^2\geq 21s^2\rightarrow(2)\text{ and so, }(1)+(2)\Rightarrow$$

$$(*)\text{ is true}\therefore\left(2\sqrt{3}-3\right)(n_a+n_b+n_c)+\left(4-2\sqrt{3}\right)(m_a+m_b+m_c)\geq\sqrt{3}s$$
$$\forall\,\Delta\,ABC,\text{''}=\text{'' iff }\Delta\,ABC\text{ is equilateral (QED)}$$