ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $g_a, g_b, g_c \to$ Gergonne cevians, the following relationship holds : $(g_a+g_b+g_c)^2 \le s^2+4(2\sqrt{5}-1)Rr+2(31-8\sqrt{5})r^2$

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$$(g_a+g_b+g_c)^2=\sum_{cyc} g_a^2+2\sum_{cyc} g_a g_b \overset{\text{Bogdan Fustei}}{=} \sum_{cyc}((s-a)^2+2rh_a)$$

$$+2.\sum_{cyc}\sqrt{\left((s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}\right)\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)}$$

$$=s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R}+$$

$$2\sum_{cyc}\left(\sqrt{(s-a)(s-b)}.\sqrt{\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\left(s-b+\frac{b^2-(c-a)^2}{b}\right)}\right)\overset{CBS}{\le}$$

$$s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R}+$$

$$2.\sqrt{\sum_{cyc}((s-a)(s-b))}.\sqrt{\sum_{cyc}\left(s^2-s.\frac{(c-a)^2}{b}-s.\frac{(b-c)^2}{a}+\frac{1}{4Rrs}.c(b-c)^2(c-a)^2\right)} \text{ and}$$

$$\because \sum_{cyc}\frac{(b-c)^2}{a}=\frac{(2R-r)s^2-r(4R+r)^2}{Rs} \therefore (g_a+g_b+g_c)^2 \le s^2-8Rr-2r^2$$

$$+\frac{r(s^2+4Rr+r^2)}{R}+$$

$$2\sqrt{r(4R+r)}.\sqrt{3s^2-\frac{2(2R-r)s^2-r(4R+r)^2}{R}+\frac{\sum_{cyc}(c(b-c)^2(c-a)^2)}{4Rrs}} \to (m)$$

$$\text{Now}\sum_{cyc}(c(b-c)^2(c-a)^2)=\sum_{cyc}\left(c\left(\sum_{cyc}a^2-2bc-a^2\right)\left(\sum_{cyc}a^2-2ca-b^2\right)\right)$$

$$=2s\left(\sum_{cyc}a^2\right)^2-(3P-16Rrs)\left(\sum_{cyc}a^2\right)+2\sum_{cyc}\left(b^2c^2(2s-a)\right)+4Rrs\left(\sum_{cyc}ab\right)$$

$$\left(\text{where } P=\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc=2s(s^2-2Rr+r^2)\right)=$$

$$8s(s^2-4Rr-r^2)^2-(12s(s^2-2Rr+r^2)-32Rrs)(s^2-4Rr-r^2)$$
$$+4s((s^2+4Rr+r^2)^2-16Rrs^2)-4Rrs(s^2+4Rr+r^2)$$
$$=4rs\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right) \text{ and so, via } (m),$$

$$(g_a+g_b+g_c)^2 \le s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R}+$$
$$2.\sqrt{r(4R+r)}.\sqrt{3s^2-2.\frac{(2R-r)s^2-r(4R+r)^2}{R}+\frac{4rs\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right)}{4Rrs}}$$
$$\overset{\text{Gerretsen}}{\le} s^2-8Rr-2r^2+\frac{4r(R+r)^2}{R}+2.\sqrt{r(4R+r)}.\sqrt{\frac{r(4R+r)(5R+8r)}{R}}$$
$$\overset{?}{\le} s^2+4(2\sqrt{5}-1)Rr+2(31-8\sqrt{5})r^2$$
$$\Leftrightarrow 4Rr+64r^2-\frac{4r(R+r)^2}{R}+8\sqrt{5}r(R-2r) \overset{?}{\ge} 2r(4R+r).\sqrt{\frac{5R+8r}{R}}$$
$$\Leftrightarrow 4\sqrt{5}.(R-2r)+\frac{2r(14R-r)}{R} \overset{?}{\ge} (4R+r).\sqrt{\frac{5R+8r}{R}}$$
$$\Leftrightarrow 80R^2(R-2r)^2+4r^2(14R-r)^2+16\sqrt{5}.Rr(R-2r)(14R-r)$$
$$\overset{?}{\ge} R(5R+8r)(4R+r)^2$$
$$\Leftrightarrow 16\sqrt{5}.Rr(R-2r)(14R-r) \boxed{\overset{?}{\ge}} r(R-2r)(488R^2-59Rr+2r^2)$$
$$\Leftrightarrow 16\sqrt{5}.R(14R-r) \overset{?}{\ge} 488R^2-59Rr+2r^2 \left(\because R-2r \overset{\text{Euler}}{\ge} 0\right)$$
$$\Leftrightarrow 1280R^2(14R-r)^2 \overset{?}{\ge} (488R^2-59Rr+2r^2)^2 \Leftrightarrow$$
$$12736R^4+19667R^3r+2077R^2r(R-2r)+R^2r^2+236Rr^2-4r^4 \overset{?}{\ge} 0$$
$$\to \text{true (strict inequality)} \because R \overset{\text{Euler}}{\ge} 2r \therefore (g_a+g_b+g_c)^2 \le$$
$$s^2+4(2\sqrt{5}-1)Rr+2(31-8\sqrt{5})r^2\ \forall\ \Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$