ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc} g_a\right)^2 \geq s^2 + 6r(h_a + h_b + h_c)$$

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$$\sum_{cyc}(c(b-c)^2(c-a)^2) = \sum_{cyc}\left(c\left(\sum_{cyc} a^2 - 2bc - a^2\right)\left(\sum_{cyc} a^2 - 2ca - b^2\right)\right) =$$

$$8s(s^2-4Rr-r^2)^2 - 12s(s^2-2Rr+r^2)(s^2-4Rr-r^2) + 32Rrs(s^2-4Rr-r^2)$$
$$+4s((s^2+4Rr+r^2)^2 - 16Rrs^2) - 4Rrs(s^2+4Rr+r^2)$$

$$\therefore \sum_{cyc}(c(b-c)^2(c-a)^2) = 4rs\left((R-2r)s^2 - r(12R^2-21Rr-6r^2)\right) \rightarrow (1)$$

$$\text{We have : } g_a^2 g_b^2 g_c^2 \overset{\text{Bogdan Fustei}}{=} \prod_{cyc}\left((s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\right) =$$

$$r^2 s\left(s^3 - \frac{1}{4Rrs}\prod_{cyc}(b-c)^2 - s^2\sum_{cyc}\frac{(b-c)^2}{a} + \frac{s}{4Rrs}\cdot\sum_{cyc}(c(b-c)^2(c-a)^2)\right)$$

$$\overset{\text{via }(1)}{=} r^2 s\left(\begin{array}{c} s^3 - \dfrac{4r^2(-s^4+(4R^2+20Rr-2r^2)s^2 - r(4R+r)^3)}{4Rrs} \\ -s^2 . \dfrac{(2R-r)s^2 - r(4R+r)^2}{Rs} + \dfrac{4rs\left((R-2r)s^2 - r(12R^2-21Rr-6r^2)\right)}{4Rr}\end{array}\right)$$

$$= \frac{r^4\left((9R+9r)s^2+(4R+r)^3\right)}{R} \Rightarrow g_a g_b g_c = r^2 . \sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}} \rightarrow (2)$$

$$\text{and } \sum_{cyc} g_a^2 g_b^2 \overset{\text{Bogdan Fustei}}{=} \sum_{cyc}\frac{\left((y+z)(x+y)+4xyz(x+y)\right)(y^2(z+x)+4xyz)}{(x+y)(y+z)(z+x)}$$

$$(x = s-a, y = s-b, z = s-c)$$

$$= \frac{\left(\prod_{cyc}(x+y)+8xyz\right)\left(\sum_{cyc} x^2y^2\right) + 32x^2y^2z^2\sum_{cyc} x + 8xyz\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2\right)}{(x+y)(y+z)(z+x)}$$

$$= \frac{(4Rrs+8r^2s)((4Rr+r^2)^2-2r^2s^2) + 32r^4s^3 + 8r^2s(4Rr+r^2)(s^2-8Rr-2r^2)}{4Rrs}$$

$$\Rightarrow \sum_{cyc} g_a^2 g_b^2 = \frac{r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R} \rightarrow (3)$$

Also, $(g_a+g_b+g_c)^2 \ge s^2+54r^2 \to (4)$ (Ineq in tri 101 by Dang Ngoc Minh)

Since $\sum_{cyc} g_a^2 = s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R} \therefore (g_a+g_b+g_c)^2 \ge s^2+$

$6r(h_a+h_b+h_c) \Leftrightarrow \sum_{cyc} g_a g_b \ge 4Rr+r^2+\frac{r(s^2+4Rr+r^2)}{R} \to (m)$

Then, $\left(\sum_{cyc} g_a g_b\right)^2 \overset{\text{via }(2),(3),(4)}{\ge} \frac{r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R}+$

$2r^2.\sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}}.\sqrt{s^2+54r^2} \overset{?}{\ge} \frac{\left(R(4Rr+r^2)+r(s^2+4Rr+r^2)\right)^2}{R^2}$

$\Leftrightarrow \left(2r^2.\sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}}.\sqrt{s^2+54r^2}\right)^2 \overset{?}{\ge} \left(\frac{r^2(s^4+2(R+r)^2s^2+r(4R+r)^3)}{R^2}\right)^2$

$\Leftrightarrow 4R^3\left((9R+9r)s^2+(4R+r)^3\right)(s^2+54r^2)-(s^4+2(R+r)^2s^2+r(4R+r)^3)$

$\boxed{\overset{?}{\underset{(\bullet)}{\ge}}}\, 0 \;\&\; \because Q = -s^6(s^2-4R^2-4Rr-3r^2)$

$-(8R^2+12Rr+7r^2)s^4(s^2-4R^2-4Rr-3r^2)$

$-r(188R^3+220R^2r+104Rr^2+27r^3)s^2(s^2-4R^2-4Rr-3r^2) \overset{\text{Gerretsen}}{\ge} 0$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \ge Q$

$\Leftrightarrow (256R^6-816R^5r-344R^4r^2-600R^3r^3-1476R^2r^4-476Rr^5-85r^6)s^2+$

$r^2(9728R^6+4224R^5r-1248R^4r^2-1064R^3r^3-240R^2r^4-24Rr^5-r^6) \boxed{\overset{?}{\underset{(\bullet\bullet)}{\ge}}}\, 0$

$\because D = 9728R^6+4224R^5r-1248R^4r^2-1064R^3r^3-240R^2r^4-24Rr^5-r^6 =$

$r^5(R-2r)(9728t^5+23680t^4+46112t^3+91160t^2+182080t+364136)$

$+728271r^6\left(t=\frac{R}{r}\right) \overset{\text{Euler}}{\ge} 728271r^6 > 0 \therefore (\bullet\bullet)$ is trivi*ally* true (strict) whenever :

$E = 256R^6-816R^5r-344R^4r^2-600R^3r^3-1476R^2r^4-476Rr^5-85r^6 \ge 0$

and when : $E<0$, then : LHS of $(\bullet\bullet) \overset{\text{Gerretsen}}{\ge} E(4R^2+4Rr+3r^2)+r^2D \overset{?}{\ge} 0$

$\Leftrightarrow 128t^8-280t^7+732t^6-250t^5-1323t^4-1334t^3-864t^2-224t-32 \overset{?}{\ge} 0$

$\Leftrightarrow (t-2)(128t^7-24t^6+684t^5+1118t^4+913t^3+492t^2+120t+16) \overset{?}{\ge} 0$

$\to$ true $\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (m)$ is true $\therefore (g_a+g_b+g_c)^2 \ge s^2+$
$6r(h_a+h_b+h_c)\ \forall\, \Delta\, ABC$," = " iff Δ ABC is equil*ateral* (QED)