ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC the following relationship holds :

$$\left(\sum_{\text{cyc}} g_a\right)^2 \geq \frac{8R+29r}{5}(h_a+h_b+h_c)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}(c(b-c)^2(c-a)^2) = 4rs\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right) \to (1)$$

$$\text{We have : } g_a^2g_b^2g_c^2 \overset{\text{Bogdan Fustei}}{=} \prod_{\text{cyc}}\left((s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\right) =$$

$$r^2s\left(s^3-\frac{1}{4Rrs}\prod_{\text{cyc}}(b-c)^2-s^2\sum_{\text{cyc}}\frac{(b-c)^2}{a}+\frac{s}{4Rrs}.\sum_{\text{cyc}}(c(b-c)^2(c-a)^2)\right)$$

$$\overset{\text{via }(1)}{=} r^2s\left(\begin{array}{c} s^3-\dfrac{4r^2(-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3)}{4Rrs} \\ -s^2.\dfrac{(2R-r)s^2-r(4R+r)^2}{Rs}+\dfrac{4rs\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right)}{4Rr}\end{array}\right)$$

$$= \frac{r^4\left((9R+9r)s^2+(4R+r)^3\right)}{R} \Rightarrow g_ag_bg_c = r^2.\sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}} \to (2)$$

$$\text{and } \sum_{\text{cyc}} g_a^2g_b^2 \overset{\text{Bogdan Fustei}}{=} \sum_{\text{cyc}}\frac{\left((y+z)(x+y)+4xyz(x+y)\right)(y^2(z+x)+4xyz)}{(x+y)(y+z)(z+x)}$$

$$(x=s-a, y=s-b, z=s-c)$$

$$= \frac{\left(\prod_{\text{cyc}}(x+y)+8xyz\right)\left(\sum_{\text{cyc}}x^2y^2\right)+32x^2y^2z^2\sum_{\text{cyc}}x+8xyz\left(\sum_{\text{cyc}}xy\right)\left(\sum_{\text{cyc}}x^2\right)}{(x+y)(y+z)(z+x)}$$

$$= \frac{(4Rrs+8r^2s)((4Rr+r^2)^2-2r^2s^2)+32r^4s^3+8r^2s(4Rr+r^2)(s^2-8Rr-2r^2)}{4Rrs}$$

$$\Rightarrow \sum_{\text{cyc}} g_a^2g_b^2 = \frac{r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R} \to (3)$$

Also, $(g_a+g_b+g_c)^2 \geq s^2+54r^2 \to (4)$ (Ineq in triangle 101 by Dang Ngoc Minh)

$$\text{Since } \sum_{\text{cyc}} g_a^2 = s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R} \therefore (g_a+g_b+g_c)^2 \geq s^2 +$$

$$6r(h_a+h_b+h_c) \Leftrightarrow \sum_{\text{cyc}} g_ag_b \geq 4Rr+r^2+\frac{r(s^2+4Rr+r^2)}{R} \to (m)$$

$$\text{Then, } \left(\sum_{cyc} g_a g_b\right)^2 \overset{\text{via } (2),(3),(4)}{\geq} \frac{r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R}+$$

$$2r^2.\sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}}.\sqrt{s^2+54r^2} \overset{?}{\geq} \frac{\left(R(4Rr+r^2)+r(s^2+4Rr+r^2)\right)^2}{R^2}$$

$$\Leftrightarrow \left(2r^2.\sqrt{\frac{(9R+9r)s^2+(4R+r)^3}{R}}.\sqrt{s^2+54r^2}\right)^2 \overset{?}{\geq} \left(\frac{r^2(s^4+2(R+r)^2s^2+r(4R+r)^3)}{R^2}\right)^2$$

$$\Leftrightarrow 4R^3\left((9R+9r)s^2+(4R+r)^3\right)(s^2+54r^2)-(s^4+2(R+r)^2s^2+r(4R+r)^3)$$

$$\boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 0 \;\&\; \because Q = -s^6(s^2-4R^2-4Rr-3r^2)$$

$$-(8R^2+12Rr+7r^2)s^4(s^2-4R^2-4Rr-3r^2)$$

$$-r(188R^3+220R^2r+104Rr^2+27r^3)s^2(s^2-4R^2-4Rr-3r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \geq Q$

$$\Leftrightarrow \left(256R^6-816R^5r-344R^4r^2-600R^3r^3-1476R^2r^4-476Rr^5-85r^6\right)s^2+$$

$$r^2\left(9728R^6+4224R^5r-1248R^4r^2-1064R^3r^3-240R^2r^4-24Rr^5-r^6\right) \boxed{\overset{?}{\underset{(\bullet\bullet)}{\geq}}} 0$$

$$\because D = 9728R^6+4224R^5r-1248R^4r^2-1064R^3r^3-240R^2r^4-24Rr^5-r^6 =$$

$$r^5(R-2r)\left(9728t^5+23680t^4+46112t^3+91160t^2+182080t+364136\right)$$

$$+728271r^6\left(t=\frac{R}{r}\right) \overset{\text{Euler}}{\geq} 728271r^6 > 0 \therefore (\bullet\bullet) \text{ is trivially true (strict) whenever :}$$

$$E = 256R^6-816R^5r-344R^4r^2-600R^3r^3-1476R^2r^4-476Rr^5-85r^6 \geq 0$$

$$\text{and when : } E < 0, \text{ then : LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} E(4R^2+4Rr+3r^2)+r^2D \overset{?}{\geq} 0$$

$$\Leftrightarrow 128t^8-280t^7+732t^6-250t^5-1323t^4-1334t^3-864t^2-224t-32 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left(128t^7-24t^6+684t^5+1118t^4+913t^3+492t^2+120t+16\right) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (m) \text{ is true} \therefore (g_a+g_b+g_c)^2 \geq s^2+$$

$$6r(h_a+h_b+h_c) \overset{?}{\geq} \frac{8R+29r}{5}(h_a+h_b+h_c) \Leftrightarrow (2R+r)s^2 \overset{?}{\geq} r(32R^2+4Rr-r^2)$$

$$\to \text{true} \because (2R+r)s^2-r(32R^2+4Rr-r^2) \overset{\text{Gerretsen}}{\geq} (2R+r)(16Rr-5r^2)$$

$$-r(32R^2+4Rr-r^2) = 2r^2(R-2r) \overset{\text{Euler}}{\geq} 0 \therefore (g_a+g_b+g_c)^2 \geq$$

$$\frac{8R+29r}{5}(h_a+h_b+h_c)\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$