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In any Δ ABC, the following relationship holds :

$$\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\le 3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)$$

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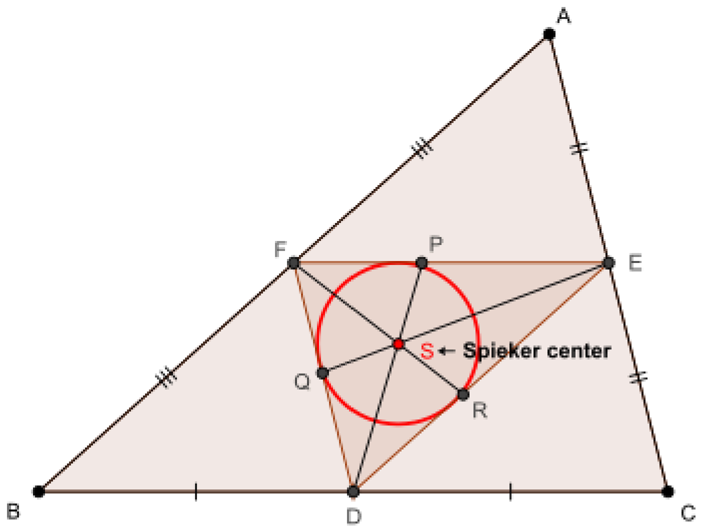

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right)-\sum_{cyc}\frac{a^4}{16}=\frac{1}{16}\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)$$

$$=\frac{16r^2s^2}{16}\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\,DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \boxed{\overset{(i)}{=}} \frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\boxed{\overset{(**)}{=}}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\mathrm{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\boxed{\overset{(\mathrm{ii})}{=}}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2\boxed{\overset{(\bullet)}{=}}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

$$\text{Now, }b^3+c^3-abc+a\left(4m_a^2\right)=b^3+c^3+a^3-abc+a\left(2b^2+2c^2-a^2\right)-a^3$$

$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2 - 6Rr - 3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3 + c^3 - abc + a(4m_a^2) \boxed{\overset{(\bullet\bullet)}{=}} 2s(s^2 - 8Rr - 3r^2 + 8Rr\cos A)$$

$$= 2s\left(s^2 - 8Rr - 3r^2 + 8Rr\left(1 - 2\sin^2\frac{A}{2}\right)\right) \therefore (\bullet), (\bullet\bullet) \Rightarrow$$

$$p_a \boxed{\overset{(\bullet\bullet\bullet)}{=}} \frac{2s}{2s+a}.\sqrt{s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}}$$

$$\text{We have}: \frac{p_a}{w_a} + \frac{p_b}{w_b} + \frac{p_c}{w_c}$$

$$= \sum_{cyc}\left(\frac{2s}{2s+a}.\sqrt{s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}}.\frac{a(b+c).\sqrt{bc}}{2abc.\sqrt{s(s-a)}}\right)$$

$$= \frac{2s}{2s(9s^2+6Rr+r^2).8Rrs}.\sum_{cyc}\left(\frac{(2s+b)(2s+c)a(b+c).\sqrt{bc(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}}.\sqrt{s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}}\right)$$

$$= \frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}.\sum_{cyc}\left(\begin{array}{c}\sqrt{(2s+b)(2s+c)a(b+c)bc(s-b)(s-c)}.\\ \sqrt{(2s+b)(2s+c)a(b+c)\left(s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}\right)}\end{array}\right)$$

$$\overset{CBS}{\leq} \frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}.\sqrt{x}.\sqrt{y}$$

$$\left(\begin{array}{c}\text{where } x = 4Rrs.\sum_{cyc}\big((2s+b)(2s+c)(b+c)(s-b)(s-c)\big) \text{ and}\\ y = \sum_{cyc}\left((2s+b)(2s+c)a(b+c)\left(s^2 - 3r^2 - 16Rr\sin^2\frac{A}{2}\right)\right)\end{array}\right)$$

$$\therefore \frac{p_a}{w_a} + \frac{p_b}{w_b} + \frac{p_c}{w_c} \boxed{\overset{(1)}{\leq}} \frac{1}{(9s^2+6Rr+r^2).8Rr^2s^2}.\sqrt{x}.\sqrt{y}$$

$$\text{Now}, \sum_{cyc}\big((2s+b)(2s+c)(s-b)(s-c)\big) = \sum_{cyc}\big((8s^2 - 2sa + bc)(s-b)(s-c)\big)$$

$$= r^2s.\sum_{cyc}\left(\frac{2s(s-a) + 6s^2 + bc}{s-a}\right) = r^2s\left(6s + \frac{6s^2(4Rr+r^2)}{r^2s} + s.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\Rightarrow \sum_{cyc}\big((2s+b)(2s+c)(s-b)(s-c)\big) \boxed{\overset{(\blacksquare)}{=}} 6s^2(4Rr+2r^2) + r^2s^2 + r^2(4R+r)^2$$

$$\textit{and also},$$

$$\sum_{cyc}\big(a(2s+b)(2s+c)(s-b)(s-c)\big) = r^2s.\sum_{cyc}\frac{a(2s(s-a)+6s^2+bc)}{s-a}$$

$$= r^2s.\left(2s(2s) + 6s^2.\sum_{cyc}\frac{a-s+s}{s-a} + \frac{4Rrs(4Rr+r^2)}{r^2s}\right)$$

$$= r^2s.\left(4s^2+6s^2.\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+4R(4R+r)\right)$$

$$\Rightarrow \sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right) \boxed{\overset{(\blacksquare\blacksquare)}{=}} r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)$$

$$\textit{and moreover}, 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$= 4Rrs.r^2s.\sum_{cyc}\frac{(2s+b)(2s+c)(s+s-a)}{s-a}$$

$$= \frac{4Rrs.r^2s^2}{r^2s}.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)+$$

$$4Rrs.r^2s.\sum_{cyc}(8s^2-2sa+bc) \overset{\text{via }(\blacksquare)}{=} 4Rrs^2\left(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2\right)$$

$$+4Rr^3s^2(21s^2+4Rr+r^2) \Rightarrow 4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(b+c)(s-b)(s-c)\right)$$

$$= x \boxed{\overset{(\blacksquare\blacksquare\blacksquare)}{=}} 8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right)$$

$$\text{Now}, y = \sum_{cyc}\left((s^2-3r^2)(2s+b)(2s+c)a(b+c)\right)$$

$$-\frac{16Rr}{4Rrs}.\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\left(\sum_{cyc}ab-bc\right)\right)$$

$$= (s^2-3r^2)\sum_{cyc}\left(\left(\sum_{cyc}ab-bc\right)(8s^2-2sa+bc)\right)-$$

$$\frac{4}{s}\left(\begin{array}{c}(s^2+4Rr+r^2).\sum_{cyc}\left(a(2s+b)(2s+c)(s-b)(s-c)\right)\\ -4Rrs.\sum_{cyc}\left((2s+b)(2s+c)(s-b)(s-c)\right)\end{array}\right)$$

$$\overset{\text{via }(\blacksquare)\text{ and }(\blacksquare\blacksquare)}{=} (s^2-3r^2)(s^2+4Rr+r^2)(21s^2+4Rr+r^2)-$$

$$(s^2-3r^2)\left(8s^2(s^2+4Rr+r^2)-24Rrs^2+(s^2+4Rr+r^2)^2-16Rrs^2\right)$$

$$-\frac{4}{s}\left(\begin{array}{c}(s^2+4Rr+r^2).r^2s\left(-8s^2+\frac{24Rs^2}{r}+16R^2+4Rr\right)\\ -4Rrs.\left(6s^2(4Rr+2r^2)+r^2s^2+r^2(4R+r)^2\right)\end{array}\right)$$

$$\Rightarrow y \boxed{\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}} 4s^2\left(3s^4-(2Rr-2r^2)s^2-r^2(16R^2+10Rr+r^2)\right)$$

$$\therefore ①, (\blacksquare\blacksquare\blacksquare), (\blacksquare\blacksquare\blacksquare\blacksquare) \Rightarrow \left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2 \le$$

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\left(\begin{array}{c}3s^4-(2Rr-2r^2)s^2\\ -r^2(16R^2+10Rr+r^2)\end{array}\right)}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4} \to (a)$$

$$\text{Now,}\left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2=9+\frac{32}{9}\cdot\frac{(R-2r)^2}{4r^2}+4\sqrt{2}\cdot\frac{R-2r}{r}$$

$$\geq 9+\frac{8(R-2r)^2}{9r^2}+\frac{28(R-2r)}{5r}\therefore\left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2$$

$$\geq\frac{40(R-2r)^2+252r(R-2r)+405r^2}{45r^2}\to(\mathrm{b})$$

$$\therefore(a),(\mathrm{b})\Rightarrow\text{it suffices to prove :}$$

$$\frac{8Rr^2s^2\left((12R+17r)s^2+r(8R^2+6Rr+r^2)\right).4s^2\begin{pmatrix}3s^4-(2Rr-2r^2)s^2\\-r^2(16R^2+10Rr+r^2)\end{pmatrix}}{(9s^2+6Rr+r^2)^2.64R^2r^4s^4}$$

$$\leq\frac{40(R-2r)^2+252r(R-2r)+405r^2}{45r^2}$$

$$\Leftrightarrow-(1620R+2295r)s^6+(6480R^3+14904R^2r+9522Rr^2-1665r^3)s^4$$
$$+r(8640R^4+30672R^3r+33948R^2r^2+9936Rr^3+675r^4)s^2$$
$$+r^2(2880R^5+13344R^4r+14600R^3r^2+5428R^2r^3+842Rr^4+4r^5)\ \boxed{\overset{?}{\geq}\ ②}\ 0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\geq 0\text{ and }s^2-(m+n)\leq 0,\text{where }m=$$
$$2R^2+10Rr-r^2\text{ and }n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$
$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\leq 0$$
$$\Rightarrow P=-(1620R+2295r)s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$-r(26676R^2+33138Rr-2925r^2)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)$$
$$\geq 0\therefore\text{in order to prove ②, it suffices to prove : LHS of ②}\geq P\Leftrightarrow$$
$$(702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4)s^2+$$
$$r(213768R^5+426828R^4r+217267R^3r^2+36170R^2r^3-140Rr^4-360r^5)\ \boxed{\overset{③}{\geq}}\ 0$$

$$\textit{and it's trivially true if : }702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4$$
$$\geq 0\textit{ and so, we now focus on the case when :}$$
$$702R^4-51345R^3r-54270R^2r^2+20484Rr^3-360r^4<0\textit{ and then :}$$

$$\text{LHS of ③}\overset{\text{Gerretsen}}{\geq}\begin{pmatrix}702R^4-51345R^3r-54270R^2r^2+\\20484Rr^3-360r^4\end{pmatrix}(4R^2+4Rr+3r^2)$$
$$+r(213768R^5+426828R^4r+217267R^3r^2+36170R^2r^3-140Rr^4-360r^5)\overset{?}{\geq}0$$
$$\Leftrightarrow 1404t^6+5598t^5+3237t^4-35956t^3-23072t^2+29936t-720\overset{?}{\geq}0$$
$$\left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow(t-2)(1404t^5+8406t^4+16352t^3+3697t(t^2-4)+4142t^2+360)\overset{?}{\geq}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow ③\Rightarrow ②\text{ is true}\Rightarrow\left(\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\right)^2\leq\left(3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\right)^2$$

$$\therefore\frac{p_a}{w_a}+\frac{p_b}{w_b}+\frac{p_c}{w_c}\leq 3+\frac{4\sqrt{2}}{3}\left(\frac{R}{2r}-1\right)\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$