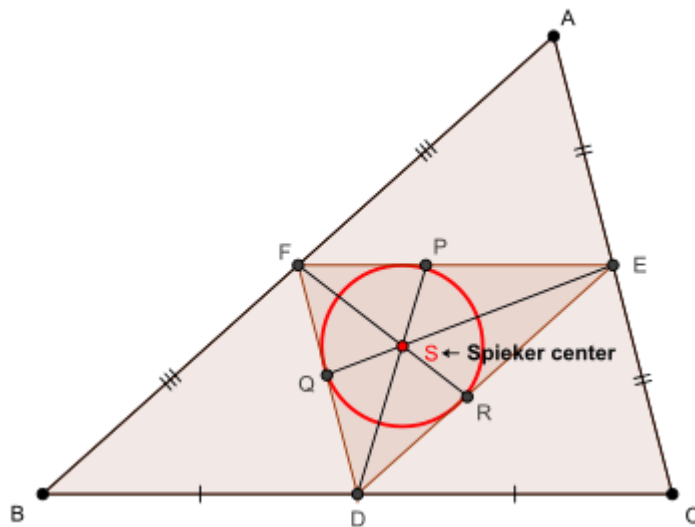


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In any acute $\triangle ABC$ with $p_a, p_b, p_c \rightarrow$ Spieker cevians,
the following relationship holds : $p_a + p_b + p_c \geq \frac{23R}{10} + \frac{22r}{5}$

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Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
and inradius of $\triangle DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4} \right) \left(\frac{b^2}{4} \right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4 \right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}}{2} \right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \triangle DEF, \therefore m(\angle AFS) = B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2}$$

$$= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on $\triangle AFS$ and $\triangle AES$, we arrive at :

$$AS^2 = \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A - B}{2}$$

$$= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A - C}{2}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4}$$

$$\begin{aligned}
 & - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\
 \text{Now, } & \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\
 & = \frac{r}{2} \left(4R \cos \frac{C}{2} \sin \frac{A-B}{2} + 4R \cos \frac{B}{2} \sin \frac{A-C}{2} \right) \\
 & = Rr \left(2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} + 2 \sin \frac{A+C}{2} \sin \frac{A-C}{2} \right) \\
 & = Rr \left(1 - 2 \sin^2 \frac{B}{2} + 1 - 2 \sin^2 \frac{C}{2} - 2 \left(1 - 2 \sin^2 \frac{A}{2} \right) \right) \\
 & = 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 & = \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 & = \frac{4(b+c)bcs \sin^2 \frac{A}{2} - 2a \cdot 2bcc \cos A}{8s} = \frac{bc \left((2s-a) \sin^2 \frac{A}{2} - a \left(1 - 2 \sin^2 \frac{A}{2} \right) \right)}{2s} \\
 & = \frac{bc \left((2s+a) \sin^2 \frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 & \Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\
 & \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 \text{Again, } & \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 & = \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}} \\
 & (i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 & = \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s} \\
 & = \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 \text{Via sine law on } \triangle AFS, & \frac{r}{2\sin\frac{C}{2} \sin \alpha} = \frac{AS}{\cos \frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}} \\
 \Rightarrow c \sin \alpha & \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b \sin \beta \stackrel{****}{=} \frac{r(a+c)}{2AS} \\
 \text{Now, } [BAX] + [BAX] &= [ABC] \Rightarrow \frac{1}{2} p_a c \sin \alpha + \frac{1}{2} p_a b \sin \beta = rs
 \end{aligned}$$

$$\text{via (***) and (***)} \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS$$

$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \stackrel{(\bullet)}{=} \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2))$$

$$\text{Now, } b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a \left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2 \right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\bullet\bullet)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right)$$

$$= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right)$$

$$\Rightarrow p_a^2 \stackrel{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\begin{aligned} \text{Now, } p_a &\stackrel{?}{\geq} h_a + \frac{2}{3} \cdot \frac{(b-c)^2}{a} \stackrel{\text{via } (\dots)}{\Leftrightarrow} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \\ &\stackrel{?}{\geq} s(s-a) - \frac{s(s-a)(b-c)^2}{a^2} + \frac{4}{9} \cdot \frac{(b-c)^4}{a^2} + \frac{4h_a}{3} \cdot \frac{(b-c)^2}{a} \\ &\Leftrightarrow \frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{a^2} - \frac{4(b-c)^2}{9a^2} \stackrel{?}{\geq} \frac{4h_a}{3a} \quad (\because (b-c)^2 \geq 0) \end{aligned}$$

$$\begin{aligned} \text{We have: } &\frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{a^2} - \frac{4(b-c)^2}{9a^2} \stackrel{\because a^2 > (b-c)^2}{>} \frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{a^2} - \frac{4}{9} \\ &= \frac{9s(3s+a)a^2 + 9s(s-a)(2s+a)^2 - 4a^2(2s+a)^2}{9a^2(2s+a)^2} \\ &= \frac{4(s-a)(9s^3 + 9s^2a + 5sa^2 + a^3)}{9a^2(2s+a)^2} \stackrel{s > a}{>} 0 \Rightarrow \text{LHS of } (\blacksquare) > 0 \therefore (\blacksquare) \Leftrightarrow \\ &\frac{T^2}{a^4(2s+a)^4} + \frac{16(b-c)^4}{81a^4} - \frac{8T(b-c)^2}{9a^4(2s+a)^2} \stackrel{?}{\geq} \frac{16}{9a^2} \cdot \left(s(s-a) - \frac{s(s-a)(b-c)^2}{a^2} \right) \\ &\quad (T = s(3s+a)a^2 + s(s-a)(2s+a)^2) \\ &\Leftrightarrow \frac{16(b-c)^4}{81a^4} - \frac{8(b-c)^2}{9a^4} \left(\frac{T}{(2s+a)^2} - 2s(s-a) \right) + \frac{T^2}{a^4(2s+a)^4} - \frac{16s(s-a)}{9a^2} \\ &\Leftrightarrow \left(\frac{4(b-c)^2}{9} \right)^2 + \frac{4(b-c)^2}{9} \cdot \frac{4s(2s^3 - 3sa^2 - a^3)}{(2s+a)^2} \\ &\quad + \frac{9T^2 - 16s(s-a)a^2(2s+a)^4}{9(2s+a)^4} \stackrel{?}{\geq} 0 \quad (\blacksquare) \end{aligned}$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in " $\frac{4(b-c)^2}{9}$ " whose discriminant

$$\begin{aligned} &= \frac{16s^2(2s^3 - 3sa^2 - a^3)^2}{(2s+a)^4} - 4 \cdot \frac{9T^2 - 16s(s-a)a^2(2s+a)^4}{9(2s+a)^4} \\ &= -\frac{16sa^7}{9(2s+a)^4} \cdot (44t^5 - 28t^4 - 49t^3 + 10t^2 + 19t + 4) \quad \left(t = \frac{s}{a} \right) \\ &= -\frac{16sa^7}{9(2s+a)^4} \cdot (t-1)^2(44t^3 + 60t^2 + 27t + 4) < 0 \Rightarrow \text{LHS of } (\blacksquare\blacksquare) > 0 \end{aligned}$$

$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a \geq h_a + \frac{2}{3} \cdot \frac{(b-c)^2}{a} \rightarrow (m)$$

$$\begin{aligned} \text{Now, } \sum_{\text{cyc}} \frac{(b-c)^2}{a} &= \sum_{\text{cyc}} \frac{b^2 + c^2 + a^2}{a} - \sum_{\text{cyc}} a - \frac{2}{4Rrs} \cdot \sum_{\text{yc}} b^2c^2 \\ &= \frac{1}{4Rrs} \left(\sum_{\text{cyc}} a^2 \right) \left(\sum_{\text{cyc}} ab \right) - \frac{8Rrs^2}{4Rrs} - \frac{2}{4Rrs} \left(\left(\sum_{\text{cyc}} ab \right)^2 - 16Rrs^2 \right) \\ &= \frac{1}{4Rrs} \left(\left(\sum_{\text{cyc}} ab \right) \left(\sum_{\text{cyc}} a^2 - 2 \sum_{\text{cyc}} ab \right) + 24Rrs^2 \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2(s^2 + 4Rr + r^2)(s^2 - 4Rr - r^2 - (s^2 + 4Rr + r^2)) + 24Rrs^2}{4Rrs} \\
 &= \frac{(2R - r)s^2 - r(4R + r)^2}{Rs} \boxed{\boxed{\boxed{\sum_{\text{cyc}} \frac{(b-c)^2}{a}}}} \\
 \text{We have : } p_a + p_b + p_c &\stackrel{\text{via (m)}}{\geq} \sum_{\text{cyc}} h_a + \frac{2}{3} \sum_{\text{cyc}} \frac{(b-c)^2}{a} \stackrel{\text{via (n)}}{=} \\
 &\quad \frac{s^2 + 4Rr + r^2}{2R} + \frac{2}{3} \cdot \frac{(2R - r)s^2 - r(4R + r)^2}{Rs} \\
 \stackrel{\text{Walker}}{\geq} &\frac{R^2 + 6Rr + 2r^2}{R} + \frac{2}{3} \cdot \frac{(2R - r)(2R^2 + 8Rr + 3r^2) - r(4R + r)^2}{Rs} \stackrel{\text{Gerretsen}}{\geq} \\
 &\quad \frac{R^2 + 6Rr + 2r^2}{R} + \frac{4}{3} \cdot \frac{2R^3 - R^2r - 5Rr^2 - 2r^3}{R \cdot \sqrt{4R^2 + 4Rr + 3r^2}} \\
 \left(\because 2R^3 - R^2r - 5Rr^2 - 2r^3 = (R - 2r)(2R^2 + 3Rr + r^2) \stackrel{\text{Euler}}{\geq} 0 \right) &\stackrel{?}{\geq} \frac{23R}{10} + \frac{22r}{5} \\
 = \frac{23R + 44r}{10} &\Leftrightarrow \frac{4}{3} \cdot \frac{2R^3 - R^2r - 5Rr^2 - 2r^3}{R \cdot \sqrt{4R^2 + 4Rr + 3r^2}} \boxed{\boxed{\boxed{\geq}}} \frac{13R^2 - 16Rr - 20r^2}{10R} \\
 \Leftrightarrow \frac{16}{9} \cdot \frac{(2R^3 - R^2r - 5Rr^2 - 2r^3)^2}{4R^2 + 4Rr + 3r^2} &\stackrel{?}{\geq} \frac{(13R^2 - 16Rr - 20r^2)^2}{100} \\
 \left(\because 13R^2 - 16Rr - 20r^2 = (13R + 10r)(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \right) & \\
 \Leftrightarrow 316t^6 + 2492t^5 - 10483t^4 + 896t^3 + 16088t^2 + 320t - 4400 &\stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \\
 \Leftrightarrow (t - 2)^2 \left(316t^4 + 3756t^3 + 2492t^2 + 510t(t - 2) + 275(t^2 - 4) \right) &\stackrel{?}{\geq} 0 \\
 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \therefore p_a + p_b + p_c &\geq \frac{23R}{10} + \frac{22r}{5} \forall \text{ acute } \triangle ABC, \\
 &\text{with equality iff } \triangle ABC \text{ is equilateral (QED)}
 \end{aligned}$$