ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c

$\to$ Nagel cevians, the following relationship holds :

$$n_a + n_b + n_c \geq m_a + m_b + m_c + \frac{1}{2}\left(s - 3\sqrt{3}r\right)$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 =$$
$$as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - a + \frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 \overset{(\bullet)}{=} s(s-a) + \frac{s}{a}.(b-c)^2$$
$$\text{Now, } n_a - m_a \overset{?}{\geq} \frac{(b-c)^2}{2a} \overset{\text{via }(\bullet)}{\Leftrightarrow} s(s-a) + \frac{s}{a}.(b-c)^2 - s(s-a) - \frac{(b-c)^2}{4}$$
$$\overset{?}{\geq} \frac{(b-c)^4}{4a^2} + m_a.\frac{(b-c)^2}{a} \Leftrightarrow \frac{4s-a}{4} - \frac{(b-c)^2}{4a} \boxed{\overset{?}{\underset{(1)}{\geq}}} m_a \; (\because (b-c)^2 \geq 0)$$
$$\text{Now, } \frac{4s-a}{4} - \frac{(b-c)^2}{4a} > \frac{4s-a}{4} - \frac{a^2}{4a} = \frac{4s-2a}{4} > 0 \therefore (1) \Leftrightarrow$$
$$\frac{(a(4s-a)-(b-c)^2)^2}{16a^2} \overset{?}{\geq} s(s-a) + \frac{(b-c)^2}{4}$$
$$\Leftrightarrow a^2(4s-a)^2 + (b-c)^4 - 2a(4s-a)(b-c)^2 \overset{?}{\geq} 16a^2s(s-a) + 4a^2(b-c)^2$$
$$\Leftrightarrow (b-c)^4 - (8sa+2a^2)(b-c)^2 + a^2(16s^2 - 8sa + a^2 - 16s^2 + 16sa) \overset{?}{\geq} 0$$
$$\Leftrightarrow (b-c)^4 - (8sa+2a^2)(b-c)^2 + a^2(8sa+a^2) \boxed{\overset{?}{\underset{(2)}{\geq}}} 0$$

Now, LHS of (2) is a quadratic polynomial in $(b-c)^2$ with discriminant $=$ $(8sa+2a^2)^2 - 4a^2(8sa+a^2) = 64a^2s^2 \therefore$ in order to prove (2),

$$\text{it suffices to prove : } (b-c)^2 \overset{?}{\leq} \frac{8sa+2a^2-\sqrt{64a^2s^2}}{2} \Leftrightarrow a^2 \overset{?}{\geq} (b-c)^2 \to \text{true}$$
$$\Rightarrow (2) \Rightarrow (1) \text{ is true} \therefore n_a - m_a \geq \frac{(b-c)^2}{2a} \text{ and analogs} \to (m)$$
$$\text{Now, } \sum_{cyc}\frac{(b-c)^2}{a} = \sum_{cyc}\frac{b^2+c^2+a^2}{a} - \sum_{cyc}a - \frac{2}{4Rrs}.\sum_{yc}b^2c^2$$

$$=\frac{1}{4Rrs}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-\frac{8Rrs^2}{4Rrs}-\frac{2}{4Rrs}\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)$$

$$=\frac{1}{4Rrs}\left(\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2-2\sum_{cyc}ab\right)+24Rrs^2\right)$$

$$=\frac{2(s^2+4Rr+r^2)\left(s^2-4Rr-r^2-(s^2+4Rr+r^2)\right)+24Rrs^2}{4Rrs}$$

$$=\frac{(2R-r)s^2-r(4R+r)^2}{Rs}\boxed{\boxed{\overset{(n)}{=}}}\sum_{cyc}\frac{(b-c)^2}{a}$$

$$\therefore\ (m)\ and\ (n)\Rightarrow\sum_{cyc}(n_a-m_a)\geq\frac{(2R-r)s^2-r(4R+r)^2}{2Rs}\overset{?}{\geq}\frac{1}{2}\left(s-3\sqrt{3}r\right)$$

$$\Leftrightarrow (R-r)s^2+3\sqrt{3}Rrs-r(4R+r)^2\boxed{\underset{(*)}{\overset{?}{\geq}}}0$$

We have, via Mitrinovic : LHS of $(*)\geq(R-r)s^2+2rs^2-r(4R+r)^2$

$$=(R+r)s^2-r(4R+r)^2\overset{\text{Gerretsen}}{\geq}(R+r)(16Rr-5r^2)-r(4R+r)^2$$

$$=3r^2(R-2r)\overset{\text{Euler}}{\geq}0\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}(n_a-m_a)\geq\frac{1}{2}\left(s-3\sqrt{3}r\right)$$

$$\therefore\ n_a+n_b+n_c\geq m_a+m_b+m_c+\frac{1}{2}\left(s-3\sqrt{3}r\right)\ \forall\,\Delta\,ABC,$$

with equality iff Δ ABC is equilateral (QED)