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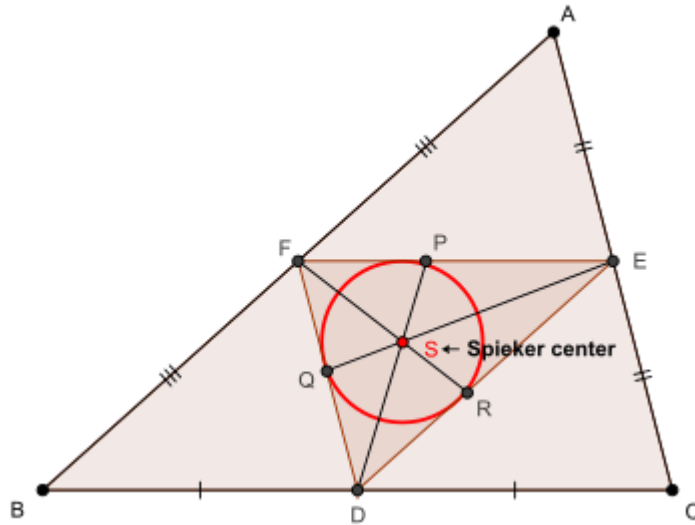
In any ΔABC with p_a, p_b, p_c

→ Spieker cevians, the following relationship holds :

$$(p_a + p_b + p_c)^2 \geq 4s^2 - \frac{9r(64R - 53r)}{25}$$

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Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4} \right) \left(\frac{b^2}{4} \right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4 \right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2} \right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\angle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A - C}{2} \end{aligned}$$

$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} + \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2}$$

$$\begin{aligned} \text{Now, } & \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} + \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\ &= \frac{r}{2} \left(4R \cos \frac{C}{2} \sin \frac{A-B}{2} + 4R \cos \frac{B}{2} \sin \frac{A-C}{2} \right) \\ &= Rr \left(2\sin \frac{A+B}{2} \sin \frac{A-B}{2} + 2\sin \frac{A+C}{2} \sin \frac{A-C}{2} \right) \\ &= Rr \left(1 - 2\sin^2 \frac{B}{2} + 1 - 2\sin^2 \frac{C}{2} - 2 \left(1 - 2\sin^2 \frac{A}{2} \right) \right) \\ &= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\ &= \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2 + c^2) - (b+c)(b-c)^2) \\ &= \frac{4(b+c)bcsin^2 \frac{A}{2} - 2a \cdot 2bccosA}{8s} = \frac{bc \left((2s-a)\sin^2 \frac{A}{2} - a \left(1 - 2\sin^2 \frac{A}{2} \right) \right)}{2s} \\ &= \frac{bc \left((2s+a)\sin^2 \frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\ &\Rightarrow - \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} - \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2} \\ &\stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \end{aligned}$$

$$\begin{aligned} \text{Again, } & \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\ &= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}} \\ (i), (*), (**) & \Rightarrow 2AS^2 = \frac{b^2 + c^2 + ab + ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\ &= \frac{(a+b+c)(b^2 + c^2 + ab + ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s} \\ &= \frac{b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3 + c^3 - abc + a(4m_a^2)}{4s} \end{aligned}$$

$$\text{Via sine law on } \triangle AFS, \frac{r}{2\sin \frac{C}{2} \sin \alpha} = \frac{AS}{\cos \frac{A-B}{2}} = \frac{cAS}{(a+b)\sin \frac{C}{2}}$$

$$\Rightarrow c \sin \alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b \sin \beta \stackrel{****}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2} p_a c \sin \alpha + \frac{1}{2} p_a b \sin \beta = rs$$

$$\stackrel{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS$$

$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 = \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2)) \Rightarrow p_a^2 - m_a^2 =$$

$$\begin{aligned} & \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2)) - m_a^2 \\ &= \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right) m_a^2 \\ &= \frac{4(a+b+c)(b^3 + c^3 - abc) - (2b^2 + 2c^2 - a^2)(b+c)^2}{4(2s+a)^2} \\ &= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2 - c^2)^2}{4(2s+a)^2} \end{aligned}$$

$$= \frac{(b-c)^2}{4(2s+a)^2} \left((a^2 + 2a(b+c) + (b+c)^2) + ((b+c)^2 + 2a(b+c) + a^2) - a^2 \right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2} (2(a+b+c)^2 - a^2) = \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \stackrel{(*)}{=} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \Leftrightarrow p_a^2 \geq m_a^2 + \frac{(b-c)^4}{4(2s+a)^2} + \frac{m_a \cdot (b-c)^2}{2s+a}$$

$$\stackrel{\text{via } (*)}{\Leftrightarrow} \frac{(b-c)^2(8s^2 - a^2 - (b-c)^2)}{4(2s+a)^2} \stackrel{(\blacksquare)}{\geq} \frac{m_a \cdot (b-c)^2}{2s+a}$$

$$\text{We note that : } 8s^2 - a^2 - (b-c)^2 > 8s^2 - 2a^2 > 0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{(8s^2 - a^2)^2 + (b-c)^4 - 2(8s^2 - a^2)(b-c)^2}{16(2s+a)^2}$$

$$\geq \left(s(s-a) + \frac{(b-c)^2}{4} \right) (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow (8s^2 - a^2)^2 - 16s(s-a)(2s+a)^2 + (b-c)^4 - 2(b-c)^2(8s^2 - a^2 + 2(2s+a)^2) \geq 0$$

$$\Leftrightarrow (b-c)^4 - 2(4s+a)^2(b-c)^2 + a^2(32s^2 + 16sa + a^2) \stackrel{(\blacksquare\blacksquare)}{\geq} 0$$

$$\text{Now, } (\blacksquare\blacksquare) \text{ is a quadrilateral in } (b-c)^2 \text{ with discriminant} =$$

$$4(4s+a)^4 - 4a^2(32s^2 + 16sa + a^2) = 256s^2(2s+a)^2 \therefore \text{in order to prove } (\blacksquare\blacksquare), \text{ it suffices to prove : } (b-c)^2 \leq \frac{2(4s+a)^2 - 16s(2s+a)}{2}$$

$$= a^2 \rightarrow \text{true} \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true} \therefore p_a - m_a \stackrel{(\bullet\bullet)}{\geq} \frac{(b-c)^2}{2(2s+a)} \text{ and analogs}$$

$$\text{Via Chu and Yang, } \left(\sum_{\text{cyc}} m_a \right)^2 \geq \frac{9s^2}{4} + \frac{7s^2}{4} - 28Rr + 29r^2 \stackrel{\text{Gerretsen}}{\geq} \frac{9s^2}{4} + \frac{7(16Rr - 5r^2)}{4} - 28Rr + 29r^2 = \frac{9s^2}{4} + \left(29 - \frac{35}{4}\right)r^2 > \frac{9s^2}{4} \Rightarrow \sum_{\text{cyc}} m_a \stackrel{(\dots)}{>} \frac{3s}{2}$$

$$\text{We have : } \sum_{\text{cyc}} \frac{(b-c)^2}{(2s+a)} \stackrel{(l)}{=} \frac{1}{2s(9s^2 + 6Rr + r^2)} \cdot \sum_{\text{cyc}} ((b-c)^2(8s^2 - 2sa + bc))$$

$$\text{and, } \sum_{\text{cyc}} ((b-c)^2(8s^2 - 2sa + bc)) = 8s^2 \sum_{\text{cyc}} (b-c)^2 - 2s \cdot \sum_{\text{cyc}} a(b^2 + c^2 - 2bc)$$

$$+ \sum_{\text{cyc}} \left(bc \left(\sum_{\text{cyc}} a^2 - a^2 - 2bc \right) \right)$$

$$= 16s^2(s^2 - 12Rr - 3r^2) - 2s \cdot \left(\left(\sum_{\text{cyc}} a \right) \left(\sum_{\text{cyc}} ab \right) - 9abc \right) +$$

$$2(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2) - 2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 8Rrs^2$$

$$= 16s^2(s^2 - 12Rr - 3r^2) - 4s^2(s^2 - 14Rr + r^2) +$$

$$2(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2) - 2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 8Rrs^2 \stackrel{\text{via (l)}}{\Rightarrow}$$

$$\sum_{\text{cyc}} \frac{(b-c)^2}{2(2s+a)} \stackrel{\boxed{(m)}}{=} \frac{4s^2(s^2 - 12Rr - 3r^2) - s^2(s^2 - 14Rr + r^2) + r((2R-r)s^2 - r(4R+r)^2)}{s(9s^2 + 6Rr + r^2)}$$

$$= \frac{\sigma}{s(9s^2 + 6Rr + r^2)} \text{ (say) } \therefore (\bullet\bullet) \text{ and (m) } \Rightarrow$$

$$p_a + p_b + p_c \geq \sum_{\text{cyc}} m_a + \frac{\sigma}{s(9s^2 + 6Rr + r^2)} \Rightarrow (p_a + p_b + p_c)^2 \geq$$

$$\left(\sum_{\text{cyc}} m_a \right)^2 + \frac{\sigma^2}{s^2(9s^2 + 6Rr + r^2)^2} + \frac{2\sigma}{s(9s^2 + 6Rr + r^2)} \cdot \left(\sum_{\text{cyc}} m_a \right)$$

via Chu and Yang, and (\bullet\bullet\bullet)

$$\geq 4s^2 - 28Rr + 29r^2 + \frac{\sigma^2}{s^2(9s^2 + 6Rr + r^2)^2} + \frac{3\sigma}{9s^2 + 6Rr + r^2}$$

$$\stackrel{?}{\geq} 4s^2 - \frac{9r(64R - 53r)}{25} \Leftrightarrow \frac{\sigma^2 + 3s^2(9s^2 + 6Rr + r^2)\sigma}{s^2(9s^2 + 6Rr + r^2)^2} \stackrel{?}{\geq} \frac{124r(R - 2r)}{25}$$

$$\Leftrightarrow 2250s^8 - (35094Rr - 8763r^2)s^6 - r^2(15392R^2 - 31652Rr - 7489r^2)s^4 +$$

$$r^3(13936R^3 + 26640R^2r + 9002Rr^2 + 873r^3)s^2 + 25r^4(4R + r)^4 \stackrel{?}{\geq} 0$$

$$\text{Now, } T = 2250(s^2 - 16Rr + 5r^2)^4 + (108906Rr - 36237r^2)(s^2 - 16Rr + 5r^2)^3$$

$$+ 2r^2(878048R^2 - 590657Rr + 106772r^2)(s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0$$

\therefore \text{ in order to prove } \textcircled{1}, \text{ it suffices to prove : LHS of } \textcircled{1} \geq T

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$$\Leftrightarrow (589575R^3 - 611967R^2r + 243365Rr^2 - 33862r^3)s^2 \stackrel{?}{\geq} \Delta_{ABC} \quad (2)$$

$$r(9433200R^4 - 12568272R^3r + 6508764R^2r^2 - 1537665Rr^3 + 138450r^4)$$

$$\text{Now, LHS of (2)} \stackrel{\text{Rouche}}{\geq} \left(\begin{matrix} 589575R^3 - 611967R^2r \\ +243365Rr^2 - 33862r^3 \end{matrix} \right) \left(\begin{matrix} 2R^2 + 10Rr - r^2 \\ -2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \end{matrix} \right)$$

$$\stackrel{?}{\geq} \text{RHS of (2)}$$

$$\Leftrightarrow (R - 2r) \left(\begin{matrix} 1179150R^4 - 2403084R^3r + 1539589R^2r^2 \\ -451693Rr^3 + 52294r^4 \end{matrix} \right)$$

$$\stackrel{?}{\geq} 2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \cdot (589575R^3 - 611967R^2r + 243365Rr^2 - 33862r^3)$$

$$\text{and } \because R - 2r \stackrel{\text{Euler}}{\geq} 2 \therefore \text{it suffices to prove :}$$

$$(1179150R^4 - 2403084R^3r + 1539589R^2r^2 - 451693Rr^3 + 52294r^4)^2 >$$

$$4(R^2 - 2Rr)(589575R^3 - 611967R^2r + 243365Rr^2 - 33862r^3)^2$$

$$\left(\begin{matrix} \because 1179150R^4 - 2403084R^3r + 1539589R^2r^2 - 451693Rr^3 + 52294r^4 \\ = (R - 2r)(1179150R^3 - 44784R^2r + 1450021Rr^2 + 2448349r^3) + 4948992r^4 \end{matrix} \right) \stackrel{\text{Euler}}{>} 0$$

$$\Leftrightarrow 986,948,550,000t^6 - 1,821,847,095,300t^5 +$$

$$(1,559,557,564,533t^4 - 770,879,282,042t^3) +$$

$$(228,609,350,325t^2 - 38,068,587,132t) + 2,734,662,436 > 0 \left(t = \frac{R}{r} \right)$$

$$\text{which is clearly true } \because t \stackrel{\text{Euler}}{\geq} 2 \text{ and hence, } 986,948,550,000t^6 \geq$$

$$2 * 986,948,550,000t^5 = 1,973,897,100,000t^5 > 1,821,847,095,300t^5$$

$$(\text{here, comma is the thousands - seperator}) \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore (p_a + p_b + p_c)^2 \geq 4s^2 - \frac{9r(64R - 53r)}{25} \forall \Delta ABC,$$

$$\text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$