# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta\, ABC$ with $n_a, n_b, n_c$**

**$\to$ Nagel's cevians, the following relationship holds :**

$$n_a + n_b + n_c + 18r \ge 3(h_a + h_b + h_c)$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2 \Rightarrow n_a^2 \overset{(1)}{=} s(s-a) + \frac{s}{a}(b-c)^2$$
$$\text{Now, } n_a \overset{?}{\ge} \frac{b^2 - bc + c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\ge} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\ge} \left(\frac{b^2-bc+c^2}{bc}\right)^2 - 1$$
$$= \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \overset{?}{\ge} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\overset{\text{via }(1)}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\ge}$$
$$\frac{(b-c)^2(b^2+c^2)}{b^2c^2} \cdot \frac{b^2c^2}{4R^2} \Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\ge} \frac{(b-c)^2(b^2+c^2)}{4R^2}$$
$$\Leftrightarrow \frac{s^2}{a^2} \overset{?}{\ge} \frac{b^2+c^2}{4R^2}\ (\because (b-c)^2 \ge 0) \Leftrightarrow 4R^2s^2 \overset{?}{\ge} a^2b^2 + c^2a^2 \to \text{true}$$
$$(\text{strict inequality}) \because 4R^2s^2 \overset{\text{Goldstone}}{\ge} \sum_{\text{cyc}} a^2b^2 > a^2b^2 + c^2a^2$$
$$\therefore n_a \ge \frac{b^2-bc+c^2}{2R} \text{ and analogs} \Rightarrow n_a + n_b + n_c + 18r - 3(h_a+h_b+h_c)$$
$$\ge \frac{2\sum_{\text{cyc}} a^2 - \sum_{\text{cyc}} ab - 3\sum_{\text{cyc}} ab}{2R} + 18r$$
$$= \frac{2(s^2 - 4Rr - r^2) - 2(s^2 + 4Rr + r^2) + 18Rr}{R} = \frac{2Rr - 4r^2}{R} = \frac{2r(R-2r)}{R} \overset{\text{Euler}}{\ge} 0$$
$$\therefore n_a + n_b + n_c + 18r \ge 3(h_a + h_b + h_c)\ \forall\, \Delta\, ABC,$$
$$\text{with equality iff } \Delta\, ABC \text{ is equilateral (QED)}$$