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In $any\ \Delta\ \mathrm{ABC}$ with $\mathrm{n}_a, \mathrm{n}_\mathrm{b}, \mathrm{n}_\mathrm{c}$

$\rightarrow$ Nagel cevians, the following relationship holds :

$$\mathrm{n}_a + \mathrm{n}_\mathrm{b} + \mathrm{n}_\mathrm{c} \geq \mathrm{m}_a + \mathrm{m}_\mathrm{b} + \mathrm{m}_\mathrm{c} + \frac{\mathrm{n}_a^2 + \mathrm{n}_\mathrm{b}^2 + \mathrm{n}_\mathrm{c}^2 - \mathrm{s}^2}{2s}$$

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$$\text{Stewart's theorem} \Rightarrow \mathrm{b}^2(\mathrm{s}-\mathrm{c}) + \mathrm{c}^2(\mathrm{s}-\mathrm{b}) = a\mathrm{n}_a^2 + a(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})$$
$$\Rightarrow \mathrm{s}(\mathrm{b}^2+\mathrm{c}^2) - \mathrm{bc}(2\mathrm{s}-a) = a\mathrm{n}_a^2 + a(s^2 - s(2\mathrm{s}-a) + \mathrm{bc}) \Rightarrow \mathrm{s}(\mathrm{b}^2+\mathrm{c}^2) - 2\mathrm{sbc}$$
$$= a\mathrm{n}_a^2 + a(a\mathrm{s} - \mathrm{s}^2) \Rightarrow \mathrm{s}(\mathrm{b}^2+\mathrm{c}^2-a^2-2\mathrm{bc}) = a\mathrm{n}_a^2 - a\mathrm{s}^2 \Rightarrow a\mathrm{n}_a^2 =$$
$$a\mathrm{s}^2 + \mathrm{s}(2\mathrm{bc}\cos\mathrm{A} - 2\mathrm{bc}) = a\mathrm{s}^2 - 4\mathrm{sbc}\sin^2\frac{\mathrm{A}}{2} = a\mathrm{s}^2 - \frac{4\mathrm{sbc}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})(\mathrm{s}-a)}{\mathrm{bc}(\mathrm{s}-a)}$$
$$= a\mathrm{s}^2 - \frac{a\mathrm{s}(\mathrm{c}+a-\mathrm{b})(a+\mathrm{b}-\mathrm{c})}{a} = a\mathrm{s}^2 - a\mathrm{s}\left(\frac{a^2-(\mathrm{b}-\mathrm{c})^2}{a}\right)$$
$$\Rightarrow \mathrm{n}_a^2 = \mathrm{s}\left(\mathrm{s} - \frac{a^2-(\mathrm{b}-\mathrm{c})^2}{a}\right) \Rightarrow \mathrm{n}_a^2 = \mathrm{s}\left(\mathrm{s} - a + \frac{(\mathrm{b}-\mathrm{c})^2}{a}\right)$$
$$\Rightarrow \mathrm{n}_a^2 \overset{(\bullet)}{=} \mathrm{s}(\mathrm{s}-a) + \frac{\mathrm{s}}{a}.(\mathrm{b}-\mathrm{c})^2$$
$$\text{Now, } \mathrm{n}_a - \mathrm{m}_a \overset{?}{\geq} \frac{(\mathrm{b}-\mathrm{c})^2}{2a} \overset{\text{via }(\bullet)}{\Leftrightarrow} \mathrm{s}(\mathrm{s}-a) + \frac{\mathrm{s}}{a}.(\mathrm{b}-\mathrm{c})^2 - \mathrm{s}(\mathrm{s}-a) - \frac{(\mathrm{b}-\mathrm{c})^2}{4}$$
$$\overset{?}{\geq} \frac{(\mathrm{b}-\mathrm{c})^4}{4a^2} + \mathrm{m}_a.\frac{(\mathrm{b}-\mathrm{c})^2}{a} \Leftrightarrow \frac{4\mathrm{s}-a}{4} - \frac{(\mathrm{b}-\mathrm{c})^2}{4a} \boxed{\overset{?}{\underset{①}{\geq}}} \mathrm{m}_a\ (\because (\mathrm{b}-\mathrm{c})^2 \geq 0)$$
$$\text{Now, } \frac{4\mathrm{s}-a}{4} - \frac{(\mathrm{b}-\mathrm{c})^2}{4a} > \frac{4\mathrm{s}-a}{4} - \frac{a^2}{4a} = \frac{4\mathrm{s}-2a}{4} > 0 \therefore ① \Leftrightarrow$$
$$\frac{(a(4\mathrm{s}-a) - (\mathrm{b}-\mathrm{c})^2)^2}{16a^2} \overset{?}{\geq} \mathrm{s}(\mathrm{s}-a) + \frac{(\mathrm{b}-\mathrm{c})^2}{4}$$
$$\Leftrightarrow a^2(4\mathrm{s}-a)^2 + (\mathrm{b}-\mathrm{c})^4 - 2a(4\mathrm{s}-a)(\mathrm{b}-\mathrm{c})^2 \overset{?}{\geq} 16a^2\mathrm{s}(\mathrm{s}-a) + 4a^2(\mathrm{b}-\mathrm{c})^2$$
$$\Leftrightarrow (\mathrm{b}-\mathrm{c})^4 - (8\mathrm{s}a + 2a^2)(\mathrm{b}-\mathrm{c})^2 + a^2(16\mathrm{s}^2 - 8\mathrm{s}a + a^2 - 16\mathrm{s}^2 + 16\mathrm{s}a) \overset{?}{\geq} 0$$
$$\Leftrightarrow (\mathrm{b}-\mathrm{c})^4 - (8\mathrm{s}a + 2a^2)(\mathrm{b}-\mathrm{c})^2 + a^2(8\mathrm{s}a + a^2) \boxed{\overset{?}{\underset{②}{\geq}}} 0$$

Now, LHS of ② is $a$ quadratic polynomial in $(\mathrm{b}-\mathrm{c})^2$ with discriminant $=$ $(8\mathrm{s}a + 2a^2)^2 - 4a^2(8\mathrm{s}a + a^2) = 64a^2\mathrm{s}^2 \therefore$ in order to prove ②,

$$\text{it suffices to prove : } (\mathrm{b}-\mathrm{c})^2 \overset{?}{\leq} \frac{8\mathrm{s}a + 2a^2 - \sqrt{64a^2\mathrm{s}^2}}{2} \Leftrightarrow a^2 \overset{?}{\geq} (\mathrm{b}-\mathrm{c})^2 \rightarrow \text{true}$$
$$\Rightarrow ② \Rightarrow ① \text{ is true} \therefore \mathrm{n}_a - \mathrm{m}_a \geq \frac{(\mathrm{b}-\mathrm{c})^2}{2a} \ \textit{and analogs} \rightarrow (l)$$

$$\text{Now, } \sum_{cyc}\frac{(b-c)^2}{a} = \sum_{cyc}\frac{b^2+c^2+a^2}{a} - \sum_{cyc} a - \frac{2}{4Rrs}.\sum_{yc} b^2c^2$$

$$= \frac{1}{4Rrs}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) - \frac{8Rrs^2}{4Rrs} - \frac{2}{4Rrs}\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)$$

$$= \frac{1}{4Rrs}\left(\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + 24Rrs^2\right)$$

$$= \frac{2(s^2+4Rr+r^2)\left(s^2-4Rr-r^2-(s^2+4Rr+r^2)\right)+24Rrs^2}{4Rrs}$$

$$= \frac{(2R-r)s^2 - r(4R+r)^2}{Rs} \boxed{\overset{(m)}{=}} \sum_{cyc}\frac{(b-c)^2}{a}$$

$$\text{As deduced earlier, } an_a^2 = as^2 - \frac{4s(s-b)(s-c)(s-a)}{(s-a)} = as^2 - a.\frac{4F^2}{a(s-a)}$$

$$= as^2 - 2a.\frac{2F}{a}.\frac{F}{s-a} \Rightarrow n_a^2 = s^2 - 2h_a r_a = s^2 - 2.\frac{2rs^2\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}$$

$$\Rightarrow n_a^2 = s^2 - \frac{rs^2}{R}.\sec^2\frac{A}{2} \text{ and analogs} \Rightarrow \frac{n_a^2+n_b^2+n_c^2-s^2}{2s}$$

$$= \frac{3s^2 - \frac{rs^2}{R}.\frac{s^2+(4R+r)^2}{s^2} - s^2}{2s} = \frac{(2R-r)s^2 - r(4R+r)^2}{2Rs} \overset{\text{via }(m)}{=}$$

$$\sum_{cyc}\frac{(b-c)^2}{2a} \overset{\text{via }(l)}{\leq} \sum_{cyc}(n_a - m_a) \therefore n_a + n_b + n_c \geq m_a + m_b + m_c$$

$$+\frac{n_a^2+n_b^2+n_c^2-s^2}{2s}\ \forall\ \Delta\ ABC, \text{with equality iff } \Delta\ ABC \text{ is equilateral (QED)}$$