ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta\ ABC$ with n_a, n_b, n_c

$\rightarrow$ Nagel cevians, the following relationship holds :

$$\frac{n_a - m_a}{h_a} + \frac{n_b - m_b}{h_b} + \frac{n_c - m_c}{h_c} \geq \frac{a^2 + b^2 + c^2 - ab - bc - ca}{2F}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 =$$
$$as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2 - (b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 = s\left(s - \frac{a^2 - (b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - a + \frac{(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 \overset{(\bullet)}{=} s(s-a) + \frac{s}{a}.(b-c)^2$$

$$\text{Now, } n_a - m_a \overset{?}{\geq} \frac{(b-c)^2}{2a} \overset{\text{via }(\bullet)}{\Leftrightarrow} s(s-a) + \frac{s}{a}.(b-c)^2 - s(s-a) - \frac{(b-c)^2}{4}$$
$$\overset{?}{\geq} \frac{(b-c)^4}{4a^2} + m_a.\frac{(b-c)^2}{a} \Leftrightarrow \frac{4s-a}{4} - \frac{(b-c)^2}{4a} \boxed{\underset{(1)}{\overset{?}{\geq}}} m_a \ (\because (b-c)^2 \geq 0)$$

$$\text{Now, } \frac{4s-a}{4} - \frac{(b-c)^2}{4a} > \frac{4s-a}{4} - \frac{a^2}{4a} = \frac{4s-2a}{4} > 0 \therefore (1) \Leftrightarrow$$
$$\frac{(a(4s-a) - (b-c)^2)^2}{16a^2} \overset{?}{\geq} s(s-a) + \frac{(b-c)^2}{4}$$
$$\Leftrightarrow a^2(4s-a)^2 + (b-c)^4 - 2a(4s-a)(b-c)^2 \overset{?}{\geq} 16a^2s(s-a) + 4a^2(b-c)^2$$
$$\Leftrightarrow (b-c)^4 - (8sa + 2a^2)(b-c)^2 + a^2(16s^2 - 8sa + a^2 - 16s^2 + 16sa) \overset{?}{\geq} 0$$
$$\Leftrightarrow (b-c)^4 - (8sa + 2a^2)(b-c)^2 + a^2(8sa + a^2) \boxed{\underset{(2)}{\overset{?}{\geq}}} 0$$

Now, LHS of (2) is a quadratic polynomial in $(b-c)^2$ with discriminant $=$
$(8sa + 2a^2)^2 - 4a^2(8sa + a^2) = 64a^2s^2 \therefore$ in order to prove (2),

$$\text{it suffices to prove : } (b-c)^2 \overset{?}{\leq} \frac{8sa + 2a^2 - \sqrt{64a^2s^2}}{2} \Leftrightarrow a^2 \overset{?}{\geq} (b-c)^2 \rightarrow \text{true}$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true} \therefore n_a - m_a \geq \frac{(b-c)^2}{2a} \Rightarrow \frac{n_a - m_a}{h_a} \geq \frac{a(b-c)^2}{2a.2F}$$

$$\Rightarrow \frac{n_a - m_a}{h_a} \geq \frac{(b-c)^2}{4F} \text{ and analogs}$$

$$\Rightarrow \frac{n_a - m_a}{h_a} + \frac{n_b - m_b}{h_b} + \frac{n_c - m_c}{h_c} \geq \sum_{cyc} \frac{(b-c)^2}{4F}$$

$$= \frac{a^2 + b^2 + c^2 - ab - bc - ca}{2F} \ \forall \Delta ABC, \text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$