ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$p_a + p_b + p_c \geq w_a + w_b + w_c + \frac{16(a^2 + b^2 + c^2 - ab - bc - ca)}{15s}$$

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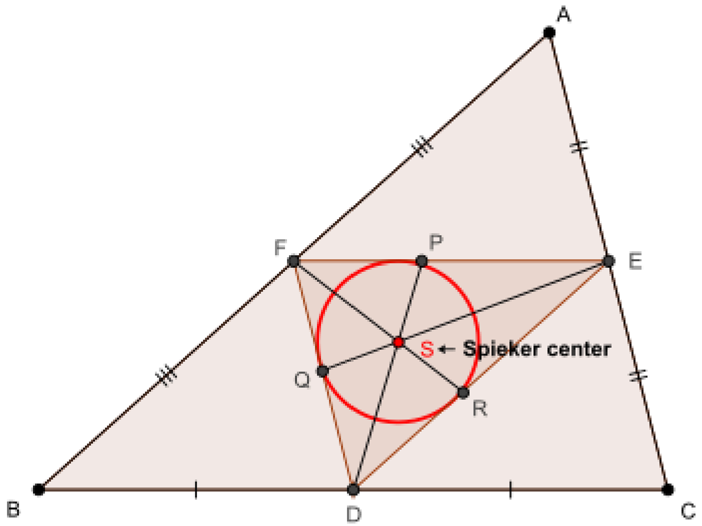

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

Via sine law on Δ AFS, $\dfrac{r}{2\sin\frac{C}{2}\sin\alpha} = \dfrac{AS}{\cos\frac{A-B}{2}} = \dfrac{cAS}{(a+b)\sin\frac{C}{2}}$

$\Rightarrow c\sin\alpha \overset{(***)}{=} \dfrac{r(a+b)}{2AS}$ and via sine law on Δ AES, $b\sin\beta \overset{(****)}{=} \dfrac{r(a+c)}{2AS}$

Now, $[BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Now, $b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$

$$= (b+c)(b^2-bc+c^2) + a(b^2-bc+c^2) + a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2) + a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a - w_a \geq \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2 - w_a^2 \geq$$

$$\frac{4s^2(b-c)^4}{(4s^2-a^2)^2}+\frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \boxed{\overset{(\blacksquare)}{\geq}} \frac{4s.w_a}{4s^2-a^2} \ (\because (b-c)^2 \geq 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} >$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2a^2}{(4s^2-a^2)^2}$$

$$= \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2} = \frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2} > 0$$

$$\therefore (\blacksquare) \Leftrightarrow \frac{\left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2(b-c)^2\right)^2}{(4s^2-a^2)^4}$$

$$\geq \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{\begin{matrix}16s^4(b-c)^4-8s^2(b-c)^2\left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)+\\ \left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)^2\end{matrix}}{(4s^2-a^2)^2}$$

$$\geq \frac{16s^2\left(s(s-a)(2s-a)^2-s(s-a)(b-c)^2\right)}{(2s-a)^2}$$

$$\Leftrightarrow 16s^4(b-c)^4-8s^2(b-c)^2\begin{pmatrix}s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\\ -2s(s-a)(2s+a)^2\end{pmatrix}$$

$$+\left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)^2-16s^3(s-a)(4s^2-a^2)^2 \geq 0$$

$$\Leftrightarrow 16s^4(b-c)^4-16s^3(b-c)^2(4s^3-4s^2a+sa^2+a^3)$$

$$+16a^2s^3(4s^3-4s^2a+a^3) \geq 0$$

$$\Leftrightarrow s(b-c)^4-(4s^3-4s^2a+sa^2+a^3)(b-c)^2+a^2(4s^3-4s^2a+a^3) \boxed{\overset{(\blacksquare\blacksquare)}{\geq}} 0$$

$$\text{and in order to prove } (\blacksquare\blacksquare)\text{, it suffices to prove :}$$

$$(b-c)^2 \leq \frac{(4s^3-4s^2a+sa^2+a^3)-\sqrt{\delta}}{2s}, \text{where } \delta =$$

$$(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3) \text{ and } \because (b-c)^2 < a^2$$

$$\therefore \text{ it suffices to prove : } 2sa^2 \le (4s^3 - 4s^2a + sa^2 + a^3)$$
$$-\sqrt{(4s^3 - 4s^2a + sa^2 + a^3)^2 - 4sa^2(4s^3 - 4s^2a + a^3)}$$
$$\Leftrightarrow \sqrt{(s-a)^2(4s^2-a^2)^2} \le 4s^3 - 4s^2a - sa^2 + a^3 = (s-a)(4s^2-a^2) \to \text{true}$$
$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - w_a \ge \frac{2s(b-c)^2}{4s^2-a^2} \text{ and analogs}$$
$$\therefore p_a + p_b + p_c \overset{(\blacksquare\blacksquare\blacksquare)}{\ge} w_a + w_b + w_c + \sum_{cyc} \frac{2s(b-c)^2}{4s^2-a^2}$$
$$\text{Now, } \sum_{cyc}\left((b-c)^2(4s^2-b^2)(4s^2-c^2)\right) =$$
$$16s^4\sum_{cyc}(b-c)^2 - 4s^2\sum_{cyc}\left((b-c)^2\left(\sum_{cyc}a^2 - a^2\right)\right)$$
$$+\sum_{cyc}\left(b^2c^2\left(\sum_{cyc}a^2 - a^2 - 2bc\right)\right)$$
$$= 32s^4(s^2-12Rr-3r^2) - 16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2) +$$
$$4s^2\sum_{cyc}\left(a^2(b^2+c^2-2bc)\right) + 2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2 - 16Rrs^2)$$
$$-48R^2r^2s^2 - 2\left((s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)\right)$$
$$= 32s^4(s^2-12Rr-3r^2) - 16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2) +$$
$$(10s^2-8Rr-2r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 64Rrs^4 - 48R^2r^2s^2$$
$$-2\left((s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)\right)$$
$$= 4(6s^6 - (64Rr+5r^2)s^4 - r^2s^2(148R^2+76Rr+12r^2) - (4Rr+r^2)^3)$$
$$\therefore \sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2} = \frac{8s\begin{pmatrix}6s^6 - (64Rr+5r^2)s^4 - r^2s^2(148R^2+76Rr+12r^2)\\ -(4Rr+r^2)^3\end{pmatrix}}{4s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)}$$
$$\overset{?}{\ge} \frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s}$$
$$\Leftrightarrow 15(6s^6 - (64Rr+5r^2)s^4 - r^2s^2(148R^2+76Rr+12r^2) - (4Rr+r^2)^3)$$
$$\overset{?}{\ge} 8(s^2-12Rr-3r^2)(9s^2+6Rr+r^2)(s^2+2Rr+r^2)$$
$$\Leftrightarrow 18s^6 - (288Rr-61r^2)s^4 - r^2s^2(12R^2-332Rr-52r^2) +$$
$$r^3(192R^3+336R^2r+108Rr^2+9r^3) \boxed{\overset{?}{\underset{(1)}{\ge}}} 0 \text{ and } \because 18(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0$$
$$\therefore \text{ in order to prove (1), it suffices to prove : LHS of (1)} \ge 18(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (576R-209r)s^4 - r(13836R^2-8972Rr+1298r^2)s^2 +$$
$$r^2(73920R^3-68784R^2r+21708Rr^2-2241r^3) \boxed{\overset{(2)}{\ge}} 0 \text{ and } \because$$

$$(576R-209r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } ②,$$
$$\text{it suffices to prove : LHS of } ② \geq (576R-209r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (1149R^2-869Rr+198r^2)s^2 \boxed{\overset{③}{\geq}}$$
$$r(18384R^3-19220R^2r+6533Rr^2-746r^3)$$

$$\text{Finally, LHS of } ③ \overset{\text{Rouche}}{\geq} (1149R^2-869Rr+198r^2)\begin{pmatrix} 2R^2+10Rr-r^2 \\ -2(R-2r).\sqrt{R^2-2Rr} \end{pmatrix}$$
$$\overset{?}{\geq} r(18384R^3-19220R^2r+6533Rr^2-746r^3)$$
$$\Leftrightarrow (R-2r)(2298R^3-4036R^2r+1705Rr^2-274r^3) \overset{?}{\geq}$$
$$2(R-2r).\sqrt{R^2-2Rr}.(1149R^2-869Rr+198r^2) \text{ and } \because (R-2r) \overset{\text{Euler}}{\geq} 0$$
$$\therefore \text{ in order to prove this, it suffices to prove :}$$
$$(2298R^3-4036R^2r+1705Rr^2-274r^3)^2$$
$$> 4(R^2-2Rr)(1149R^2-869Rr+198r^2)^2$$
$$\Leftrightarrow 3309120t^4-3964248t^3+2208945t^2-620708t+75076>0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow 1326996t^4+1982124t^3(t-2)+1898591t^2+310354t(t-2)+75076>0$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sum_{\text{cyc}} \frac{2s(b-c)^2}{4s^2-a^2} \geq$$
$$\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s} \overset{\text{via } (\blacksquare\blacksquare\blacksquare)}{\Rightarrow} p_a+p_b+p_c \geq$$
$$w_a+w_b+w_c+\frac{16(a^2+b^2+c^2-ab-bc-ca)}{15s} \forall \Delta ABC,$$
$$\text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$