ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta\, ABC$ with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$p_a + p_b + p_c \geq w_a + w_b + w_c + \frac{4s(R-2r)}{15R}$$

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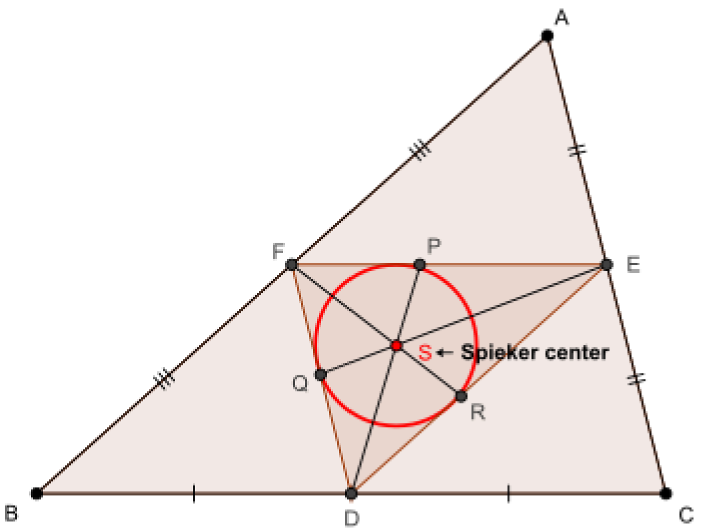

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\, DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on $\Delta\, AFS$ and $\Delta\, AES$, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

Now, $\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

Again, $\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\;(a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore\boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a - w_a \geq \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2 - w_a^2 \geq$$

$$\frac{4s^2(b-c)^4}{(4s^2-a^2)^2}+\frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \boxed{\overset{(\blacksquare)}{\geq}} \frac{4s.w_a}{4s^2-a^2} \quad (\because (b-c)^2 \geq 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2(b-c)^2}{(4s^2-a^2)^2} >$$

$$\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4s^2a^2}{(4s^2-a^2)^2}$$

$$= \frac{s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2a^2}{(4s^2-a^2)^2} = \frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2} > 0$$

$$\therefore (\blacksquare) \Leftrightarrow \frac{(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2-4s^2(b-c)^2)^2}{(4s^2-a^2)^4}$$

$$\geq \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{\begin{array}{c}16s^4(b-c)^4-8s^2(b-c)^2(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)+\\(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2\end{array}}{(4s^2-a^2)^2}$$

$$\geq \frac{16s^2(s(s-a)(2s-a)^2-s(s-a)(b-c)^2)}{(2s-a)^2}$$

$$\Leftrightarrow 16s^4(b-c)^4-8s^2(b-c)^2\left(\begin{array}{c}s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\\-2s(s-a)(2s+a)^2\end{array}\right)$$

$$+(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2)^2-16s^3(s-a)(4s^2-a^2)^2 \geq 0$$

$$\Leftrightarrow 16s^4(b-c)^4-16s^3(b-c)^2(4s^3-4s^2a+sa^2+a^3)$$

$$+16a^2s^3(4s^3-4s^2a+a^3) \geq 0$$

$$\Leftrightarrow s(b-c)^4-(4s^3-4s^2a+sa^2+a^3)(b-c)^2+a^2(4s^3-4s^2a+a^3) \boxed{\overset{(\blacksquare\blacksquare)}{\geq}} 0$$

*a*nd in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :

$$(b-c)^2 \le \frac{(4s^3-4s^2a+sa^2+a^3)-\sqrt{\delta}}{2s}, \text{where } \delta =$$
$$(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3) \text{ and } \because (b-c)^2 < a^2$$
$$\therefore \text{ it suffices to prove : } 2sa^2 \le (4s^3-4s^2a+sa^2+a^3)$$
$$-\sqrt{(4s^3-4s^2a+sa^2+a^3)^2-4sa^2(4s^3-4s^2a+a^3)}$$
$$\Leftrightarrow \sqrt{(s-a)^2(4s^2-a^2)^2} \le 4s^3-4s^2a-sa^2+a^3 = (s-a)(4s^2-a^2) \rightarrow \text{true}$$
$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - w_a \ge \frac{2s(b-c)^2}{4s^2-a^2} \text{ and analogs}$$
$$\therefore p_a+p_b+p_c \overset{(\blacksquare\blacksquare\blacksquare)}{\ge} w_a+w_b+w_c+\sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2}$$
$$\text{Now, } \sum_{cyc}\left((b-c)^2(4s^2-b^2)(4s^2-c^2)\right) =$$
$$16s^4\sum_{cyc}(b-c)^2-4s^2\sum_{cyc}\left((b-c)^2\left(\sum_{cyc}a^2-a^2\right)\right)$$
$$+\sum_{cyc}\left(b^2c^2\left(\sum_{cyc}a^2-a^2-2bc\right)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$4s^2\sum_{cyc}\left(a^2(b^2+c^2-2bc)\right)+2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)$$
$$-48R^2r^2s^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=32s^4(s^2-12Rr-3r^2)-16s^2(s^2-4Rr-r^2)(s^2-12Rr-3r^2)+$$
$$(10s^2-8Rr-2r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-64Rrs^4-48R^2r^2s^2$$
$$-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)$$
$$=4(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$
$$\therefore \sum_{cyc}\frac{2s(b-c)^2}{4s^2-a^2} =$$
$$\frac{8s(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)}{4s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)} \overset{?}{\ge} \frac{4s(R-2r)}{15R}$$
$$\Leftrightarrow 15R(6s^6-(64Rr+5r^2)s^4-r^2s^2(148R^2+76Rr+12r^2)-(4Rr+r^2)^3)$$
$$\overset{?}{\ge} 2s^2(9s^2+6Rr+r^2)(s^2+2Rr+r^2)(R-2r)$$
$$\Leftrightarrow (72R+36r)s^6-r(1008R^2-Rr-40r^2)s^4-$$
$$r^2(2244R^3+1108R^2r+150Rr^2-4r^3)s^2-15Rr^3(4R+r)^3 \boxed{\underset{①}{\overset{?}{\ge}}} 0 \text{ and } \because$$
$$(72R+36r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } ①, \text{it suffices}$$

$$\text{to prove : LHS of } ① \geq (72R + 36r)(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow$$

$$(2448R^2 + 649Rr - 500r^2)s^4$$

$$-r(57540R^3 - 5804R^2r - 11730Rr^2 + 2696r^3)s^2 +$$

$$r^2(293952R^4 - 129744R^3r - 52020R^2r^2 + 34185Rr^3 - 4500r^4) \boxed{\overset{②}{\geq}} 0 \text{ and}$$

$$\because (2448R^2 + 649Rr - 500r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } ②, \text{ it suffices to prove : LHS of } ② \geq$$

$$(2448R^2 + 649Rr - 500r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (5199R^3 + 523R^2r - 2690Rr^2 + 576r^3)s^2 \boxed{\overset{③}{\geq}}$$

$$r(293952R^4 - 129744R^3r - 52020R^2r^2 + 34185Rr^3 - 4500r^4)$$

$$\text{Finally, LHS of } ③ \overset{\text{Gerretsen}}{\geq} (5199R^3 + 523R^2r - 2690Rr^2 + 576r^3)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(293952R^4 - 129744R^3r - 52020R^2r^2 + 34185Rr^3 - 4500r^4)$$

$$\Leftrightarrow 6321t^3 - 16000t^2 + 7156t - 880 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4642t^2 + 1679t(t-2) + 440) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ③ \Rightarrow ②$$

$$\Rightarrow ① \text{ is true} \therefore \sum_{\text{cyc}} \frac{2s(b-c)^2}{4s^2 - a^2} \geq \frac{4s(R-2r)}{15R} \overset{\text{via } (\blacksquare\blacksquare\blacksquare)}{\Rightarrow}$$

$$p_a + p_b + p_c \geq w_a + w_b + w_c + \frac{4s(R-2r)}{15R} \;\forall\, \Delta\, ABC,$$

with equality iff Δ ABC is equilateral (QED)