ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $p_a \to$ Spieker cevian, the following relationship holds :

$$\frac{(b-c)^2}{2(2s+a)} \le p_a - m_a \le \frac{a|b-c|}{2(2s+a)}$$

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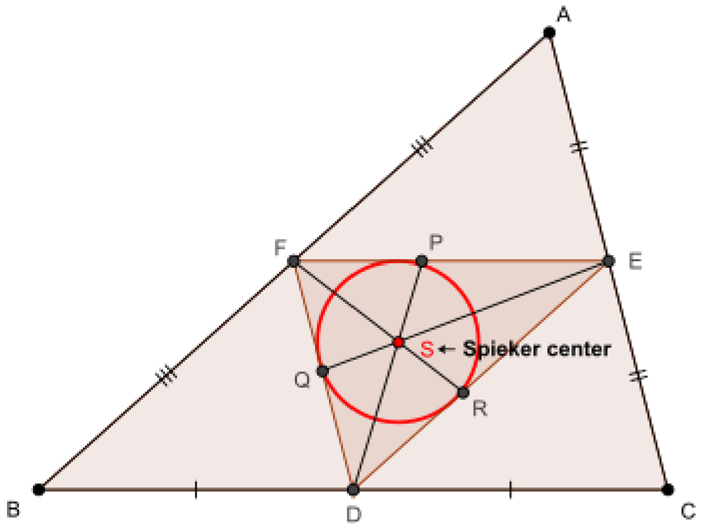

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,} \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\Rightarrow p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, } p_a - m_a \le \frac{a|b-c|}{2(2s+a)} \Leftrightarrow p_a^2 - m_a^2 \le \frac{a^2(b-c)^2}{4(2s+a)^2} + \frac{am_a.|b-c|}{2s+a}$$

$$\overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-2a^2)}{4(2s+a)^2} \overset{(\blacksquare)}{\le} \frac{am_a.|b-c|}{2s+a}$$

$$\text{Now, } a^2(4m_a^2) = a^2(2b^2+2c^2-a^2)$$

$$= 2a^2b^2+2c^2a^2+2b^2c^2-a^4-b^4-c^4+(b^4+c^4-2b^2c^2)$$

$$= 16F^2+(b^2-c^2)^2 > (b^2-c^2)^2 \Rightarrow 2am_a > \left|b^2-c^2\right|$$

$$\therefore \frac{am_a.|b-c|}{2s+a} \ge \frac{(b-c)^2(2s-a)}{2(2s+a)} = \frac{2(b-c)^2(2s-a)(2s+a)}{4(2s+a)^2}$$

$$= \frac{(b-c)^2(8s^2-2a^2)}{4(2s+a)^2} \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - m_a \le \frac{a|b-c|}{2(2s+a)}$$

$$\text{Again, } p_a - m_a \ge \frac{(b-c)^2}{2(2s+a)} \Leftrightarrow p_a^2 \ge m_a^2 + \frac{(b-c)^4}{4(2s+a)^2} + \frac{m_a.(b-c)^2}{2s+a}$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2-(b-c)^2)}{4(2s+a)^2} \overset{(\blacksquare\blacksquare)}{\geq} \frac{m_a.(b-c)^2}{2s+a}$$

We note that : $8s^2-a^2-(b-c)^2 > 8s^2-2a^2 > 0 \therefore (\blacksquare\blacksquare) \Leftrightarrow$

$$\frac{(8s^2-a^2)^2+(b-c)^4-2(8s^2-a^2)(b-c)^2}{16(2s+a)^2}$$

$$\geq \left(s(s-a)+\frac{(b-c)^2}{4}\right) \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow (8s^2-a^2)^2-16s(s-a)(2s+a)^2+(b-c)^4$$

$$-2(b-c)^2(8s^2-a^2+2(2s+a)^2) \geq 0$$

$$\Leftrightarrow (b-c)^4-2(4s+a)^2(b-c)^2+a^2(32s^2+16sa+a^2) \overset{(\blacksquare\blacksquare\blacksquare)}{\geq} 0$$

Now, $(\blacksquare\blacksquare\blacksquare)$ is a quadrilateral in $(b-c)^2$ with discriminant $=$

$$4(4s+a)^4-4a^2(32s^2+16sa+a^2) = 256s^2(2s+a)^2$$

$\therefore$ in order to prove $(\blacksquare\blacksquare\blacksquare)$, it suffices to prove :

$$(b-c)^2 \leq \frac{2(4s+a)^2-16s(2s+a)}{2} = a^2 \rightarrow \text{true} \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \text{ is true}$$

$$\therefore p_a - m_a \geq \frac{(b-c)^2}{2(2s+a)} \text{ and so, } \frac{(b-c)^2}{2(2s+a)} \leq p_a - m_a \leq \frac{a|b-c|}{2(2s+a)} \ \forall \Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)