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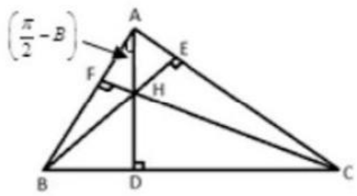
In acute $\Delta ABC, H$ –orthocenter, the following relationship holds:

$$\sec A + \sec B + \sec C \le \frac{AH}{HD} + \frac{BH}{HE} + \frac{CH}{HF}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India



$$From\ \Delta\ AFC, AF = b\cos A\,, In\ \Delta AFH, \cos\left(\frac{\pi}{2} - B\right) = \frac{AF}{AH} = \frac{b\cos A}{AH}\ or,$$
$$AH = \frac{b\cos A}{\sin B} = 2R\cos A\,,$$

$$similarly\ CH = 2R\cos C\ and\ BH = 2R\cos B\,,$$

$$In\ \Delta AFH\,, \tan\left(\frac{\pi}{2} - B\right) = \frac{FH}{AF} = \frac{FH}{b\cos A}\ or, FH = b\cos A\cot B$$

$$\frac{AH}{HD} + \frac{BH}{HE} + \frac{CH}{HF} = \sum \frac{CH}{HF} = \sum \frac{2R\cos C}{b\cos A\cot B} = \sum \frac{2R\cos C\sin B}{2R\sin\ B\cos A\cos B} =$$

$$= \sum \frac{\cos^2 C}{\cos A\cos B\cos C} \overset{AM-GM}{\ge} \frac{\sum \cos A\cos B}{\cos A\cos B\cos C} =$$

$$= \sum \frac{\cos A\cos B}{\cos A\cos B\cos C} = \sum \sec C = \sec A + \sec B + \sec C$$