ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, for $k = \dfrac{\sqrt[4]{3}\left(373\sqrt{3}+1323\right)}{1794}$, prove that:

$$\frac{1}{R+s}+\frac{1}{s+r}+\frac{1}{r+R}\le\frac{k}{\sqrt{F}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$R+s=\sqrt{R.R}+\sqrt{s.s}\overset{Mitrinovic\ \&\ Euler}{\ge}\sqrt{\frac{2s}{3\sqrt{3}}.2r}+\sqrt{s.3\sqrt{3}r}=\frac{\left(2+3\sqrt{3}\right)\sqrt{F}}{\sqrt[4]{27}}.$$

$$s+r=\frac{s}{3\sqrt{3}}+r+\frac{3\sqrt{3}-1}{3\sqrt{3}}\sqrt{s.s}\overset{AM-GM\ \&\ Mitrinovic}{\ge}2\sqrt{\frac{sr}{3\sqrt{3}}}+\frac{3\sqrt{3}-1}{3\sqrt{3}}\sqrt{s.3\sqrt{3}rs}=\frac{\left(1+3\sqrt{3}\right)\sqrt{F}}{\sqrt[4]{27}}.$$

$$r+R\overset{Mitrinovic}{\ge}r+\frac{2s}{3\sqrt{3}}=r+\frac{s}{3\sqrt{3}}+\sqrt{\frac{s.s}{27}}\overset{AM-GM\ \&\ Mitrinovic}{\ge}2\sqrt{\frac{sr}{3\sqrt{3}}}+\sqrt{\frac{sr}{3\sqrt{3}}}=\sqrt{\sqrt{3}F}.$$

Then

$$\frac{1}{R+s}+\frac{1}{s+r}+\frac{1}{r+R}\le\frac{\sqrt[4]{27}}{\left(2+3\sqrt{3}\right)\sqrt{F}}+\frac{\sqrt[4]{27}}{\left(1+3\sqrt{3}\right)\sqrt{F}}+\frac{1}{\sqrt{\sqrt{3}F}}=\frac{k}{\sqrt{F}}.$$

Equality holds iff ΔABC is equilateral.