ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, b^2+bc+c^2 \le a^2$. Prove that :

$$\frac{9r}{4R} < \frac{h_a+h_b+h_c}{r_a+r_b+r_c} < \frac{18}{25}$$

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$$\frac{9r}{4R} < \frac{h_a+h_b+h_c}{r_a+r_b+r_c} \Leftrightarrow \frac{9r}{4R} < \frac{ab+bc+ca}{2R(4R+r)} \Leftrightarrow 9.4r(4R+r) < 8(ab+bc+ca)$$

$$\Leftrightarrow 9\left[2(ab+bc+ca)-\left(a^2+b^2+c^2\right)\right] < 8(ab+bc+ca)$$

$$\Leftrightarrow 0 < 5(b+c-a)^2+4\left(a^2-b^2-bc-c^2\right)+8(b-c)^2,$$

which is true. $(\because b+c-a>0)$

$$\frac{h_a+h_b+h_c}{r_a+r_b+r_c} < \frac{18}{25} \Leftrightarrow 25\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) < 18\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)$$

$$\overset{CBS}{\Leftarrow} 25\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) < 18\left(\frac{1}{b+c-a}+\frac{4}{2a}\right) \Leftrightarrow 25\left(\frac{1}{b}+\frac{1}{c}\right) < \frac{18}{b+c-a}+\frac{11}{a}.$$

Since $bc \ge (b+c)^2-a^2 > 0$, **it suffices to prove that**

$$\frac{25(b+c)}{(b+c)^2-a^2} < \frac{18}{b+c-a}+\frac{11}{a} \overset{x:=\frac{b+c}{a}>1}{\Leftrightarrow} \frac{25x}{x^2-1} < \frac{18}{x-1}+11 \Leftrightarrow 0 < 11x^2-7x+7,$$

which is true for $x>1$. **So the proof is complete.**