ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC, for $\alpha \leq \dfrac{17+15\sqrt{3}}{5}$ and $\beta = (2+3\sqrt{3}+\alpha).3^{\frac{-3}{4}}$. Prove that:

$$R + s + \alpha r \geq \beta\sqrt{F}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is equivalent to

$$R + s \geq (2+3\sqrt{3})\sqrt{\frac{F}{3\sqrt{3}}} + \alpha\left(\sqrt{\frac{F}{3\sqrt{3}}} - r\right)$$

Since $s \geq 3\sqrt{3}r$, then $\sqrt{\dfrac{F}{3\sqrt{3}}} - r \geq 0$, so it suffices to prove that

$$R + s \geq (2+3\sqrt{3})\sqrt{\frac{F}{3\sqrt{3}}} + \frac{17+15\sqrt{3}}{5}.\left(\sqrt{\frac{F}{3\sqrt{3}}} - r\right) \text{ or}$$

$$R + \left(\sqrt{s} - \sqrt{3\sqrt{3}r}\right)^2 + \frac{17}{5}r \geq \frac{27}{5}\sqrt{\frac{F}{3\sqrt{3}}}.$$

$$\text{or } R + \frac{17}{5}r \geq \frac{27}{5}\sqrt{\frac{F}{3\sqrt{3}}} \text{ or } (5R+17r)^4 \geq 27^3 s^2 r^2.$$

By Gerretsen's inequality, we have

$$27^3 s^2 r^2 \leq$$

$$\leq 27^3(4R^2+4Rr+3r^2)r^2 = (5R+17r)^4 - (R-2r)^2(625R^2+11000Rr+6118r^2) \leq$$

$$\leq (5R+17r)^4.$$

Equality holds iff ΔABC is equilateral.