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In any ΔABC, the following relationship holds:

$$\frac{m_a w_a}{r_a h_a} \geq \frac{r}{R - r + \sqrt{R(R-2r)}}$$

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Using the known inequality $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, we have:

$$\frac{m_a w_a}{r_a h_a} \geq \frac{bc\cos^2\frac{A}{2}}{r_a h_a} = \frac{s(s-a).(s-a)a}{2F^2} = \frac{(s-a)^2 a}{2sr^2}.$$

So it suffices to prove that :

$$\frac{(s-a)^2 a}{2sr^2} \geq \frac{r}{R - r + \sqrt{R(R-2r)}} = \frac{2r}{\left(\sqrt{R} + \sqrt{R-2r}\right)^2}$$

$$\Leftrightarrow 2\sqrt{sr}\left(\sqrt{R} + \sqrt{R-2r}\right) \geq \frac{2\sqrt{sr}.2\sqrt{sr^3}}{(s-a)\sqrt{a}} \Leftrightarrow \sqrt{abc} + \sqrt{abc - 8sr^2} \geq \frac{4sr^2}{(s-a)\sqrt{a}}$$

$$\Leftrightarrow \sqrt{abc - 8sr^2} \geq \frac{4sr^2}{(s-a)\sqrt{a}} - \sqrt{abc}. \quad (1)$$

If $RHS_{(1)} \leq 0$, the inequality (1) is true. Assume now that $RHS_{(1)} \geq 0$. We have

$$(1) \Leftrightarrow -8sr^2 \geq \frac{16\left(sr^2\right)^2}{a(s-a)^2} - \frac{8sr^2\sqrt{bc}}{(s-a)} \Leftrightarrow \sqrt{bc} \geq \frac{2sr^2}{a(s-a)} + s - a = \frac{2(s-b)(s-c)}{a} + s - a.$$

By CBS and $\mathrm{HM}-\mathrm{GM}$ inequalities, we have

$$\sqrt{bc} = \sqrt{[(s-c)+(s-a)][(s-b)+(s-a)]} \geq \sqrt{(s-b)(s-c)} + s - a =$$

$$\geq \frac{2(s-b)(s-c)}{(s-b)+(s-c)} + s - a = \frac{2(s-b)(s-c)}{a} + s - a$$

which completes the proof. Equality holds iff $b = c \leq a$.