ROMANIAN MATHEMATICAL MAGAZINE

In any acute ΔABC the following relationship holds :

$$\prod_{cyc}\left(\frac{1}{r_br_c - rr_a} + \frac{1}{r_cr_a - rr_b} - \frac{1}{r_ar_b + rr_c}\right) \geq \frac{27}{(abc)^2}$$

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$$r_br_c - rr_a = \frac{F^2}{(s-b)(s-c)} - \frac{r.sr}{s-a} = s(s-a) - (s-b)(s-c) =$$
$$= \frac{(b+c)^2 - a^2}{4} - \frac{a^2 - (b-c)^2}{4} = \frac{b^2 + c^2 - a^2}{2}$$
$$r_br_c + rr_a = \frac{F^2}{(s-b)(s-c)} + \frac{r.sr}{s-a} = s(s-a) + (s-b)(s-c) =$$
$$= 2s^2 - s(a+b+c) + bc = bc$$

Let $2x := b^2 + c^2 - a^2 \geq 0,\ \ 2y := c^2 + a^2 - b^2 \geq 0,\ \ 2z := a^2 + b^2 - c^2 \geq 0$

We have

$$\frac{1}{r_br_c - rr_a} + \frac{1}{r_cr_a - rr_b} - \frac{1}{r_ar_b + rr_c} = \frac{2}{b^2 + c^2 - a^2} + \frac{2}{c^2 + a^2 - b^2} - \frac{1}{ab} =$$
$$= \frac{1}{x} + \frac{1}{y} - \frac{1}{\sqrt{(z+x)(z+y)}} =$$
$$= \frac{(x+y)\sqrt{(z+x)(z+y)} - xy}{xy\sqrt{(z+x)(z+y)}} \overset{AM-GM\ \&\ CBS}{\geq} \frac{2\sqrt{xy}.\left(z + \sqrt{xy}\right) - xy}{xy\sqrt{(z+x)(z+y)}} =$$
$$= \frac{2z + \sqrt{xy}}{\sqrt{xy(z+x)(z+y)}} \overset{AM-GM}{\geq} \frac{3\sqrt[3]{z^2\sqrt{xy}}}{\sqrt{xy(z+x)(z+y)}}$$

Therefore

$$\prod_{cyc}\left(\frac{1}{r_br_c - rr_a} + \frac{1}{r_cr_a - rr_b} - \frac{1}{r_ar_b + rr_c}\right) \geq \prod_{cyc}\frac{3\sqrt[3]{z^2\sqrt{xy}}}{\sqrt{xy(z+x)(z+y)}} =$$
$$= \frac{27}{(x+y)(y+z)(z+x)} = \frac{27}{(abc)^2}$$

Equality holds iff ΔABC is equilateral.