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In any Δ ABC the following relationship holds :

$$(R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right) \geq \frac{33+29\sqrt{3}}{6}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18} \overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq} \frac{2}{3\sqrt{3}}+\frac{\sqrt{3(4Rr+r^2)}}{r}-\frac{29\sqrt{3}}{9}$$

$$= \frac{2}{3\sqrt{3}}+\frac{3.\sqrt{4Rr+r^2}}{\sqrt{3}r}-\frac{29}{3\sqrt{3}} = \frac{9.\sqrt{4Rr+r^2}-27r}{3\sqrt{3}r} = \frac{3.\sqrt{4Rr+r^2}-9r}{\sqrt{3}r} \overset{?}{\geq} \frac{1}{2}-\frac{r}{R}$$

$$\Leftrightarrow \frac{9(4Rr+r^2)+81r^2-54r.\sqrt{4Rr+r^2}}{3r^2} \overset{?}{\geq} \frac{(R-2r)^2}{4R^2}$$

$$\Leftrightarrow \frac{3(4R+r)+27r}{r}-\frac{(R-2r)^2}{4R^2} \overset{?}{\geq} \frac{18.\sqrt{4Rr+r^2}}{r}$$

$$\Leftrightarrow \frac{\left(4R^2(12R+30r)-r(R-2r)^2\right)^2}{16R^4r^2} \overset{?}{\geq} \frac{324(4Rr+r^2)}{r^2}$$

$$\left(\because \frac{3(4R+r)+27r}{r} \overset{\text{Euler}}{\geq} 54 > 1 \overset{\text{Euler}}{\geq} \frac{(R-2r)^2}{4R^2} \Rightarrow 4R^2(12R+30r)-r(R-2r)^2 > 0\right)$$

$$\Leftrightarrow 2304t^6-9312t^5+9361t^4+568t^3-936t^2-32t+16 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(2304t^3+4512t^2+8785t+17566)+35136\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18} \geq \frac{1}{2}-\frac{r}{R} \Rightarrow \frac{R}{s}+\frac{s}{r}+\frac{r}{R} \geq \frac{9+58\sqrt{3}}{18} \to \text{(i)}$$

$$\text{Again}, \frac{r}{s}+\frac{s}{R} \overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq} \frac{2r}{3\sqrt{3}R}+\frac{\sqrt{3(4Rr+r^2)}}{R} = \frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}$$

$$\therefore \text{ in order to prove : } \frac{r}{s}+\frac{s}{R}+\frac{R}{r} \geq \frac{36+29\sqrt{3}}{18}, \text{ it suffices to prove :}$$

$$\frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}+\frac{R}{r} \geq \frac{36+29\sqrt{3}}{18} \Leftrightarrow \frac{R}{r}-2 \geq \sqrt{3}\left(\frac{29}{18}-\frac{2r}{9R}-\sqrt{\frac{4r}{R}+\frac{r^2}{R^2}}\right)$$

$$\Leftrightarrow t-2 \overset{①}{\geq} \sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right) \quad \left(t=\frac{R}{r}\right)$$

$$\text{Let } f(t) = t-2-\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right) \ \forall\, t \geq 2 \text{ and then :}$$

$$f'(t)=1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{\sqrt{\frac{4}{t}+\frac{1}{t^2}}}\right)\overset{\frac{1}{t^2}>0}{>}1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{2\sqrt{\frac{1}{t}}}\right)$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{\frac{2}{m^4}+\frac{1}{m^6}}{\frac{2}{m}}\right)\ (m=\sqrt{t})$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{1}{m^3}+\frac{1}{2m^5}\right)\overset{m=\sqrt{\frac{R}{r}}\geq\sqrt{2}\text{ via Euler}}{\geq}1-\sqrt{3}\left(\frac{2}{9.4}+\frac{1}{2\sqrt{2}}+\frac{1}{8\sqrt{2}}\right)$$

$$\approx 0.1383>0\Rightarrow f'(t)>0\ \forall\, t\geq 2\Rightarrow f(t)\text{ is }\uparrow\text{ on }[2,\infty)\Rightarrow f(t)\geq f(2)=0\ \forall\, t\geq 2$$

$$\therefore t-2\geq\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\Rightarrow (1)\text{ is true }\therefore \frac{r}{s}+\frac{s}{R}+\frac{R}{r}\geq\frac{36+29\sqrt{3}}{18}\rightarrow\text{(ii)}$$

$$\text{We have}:(R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right)=3+\frac{R}{s}+\frac{R}{r}+\frac{s}{R}+\frac{s}{r}+\frac{r}{R}+\frac{r}{s}$$

$$=3+\frac{R}{s}+\frac{s}{r}+\frac{r}{R}+\frac{r}{s}+\frac{s}{R}+\frac{R}{r}\overset{\text{via (i) + (ii)}}{\geq}3+\frac{9+58\sqrt{3}}{18}+\frac{36+29\sqrt{3}}{18}$$

$$=\frac{33+29\sqrt{3}}{6}\ \forall\,\Delta\, ABC,''=''\text{ iff }\Delta\, ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$(R+s+r)\left(\frac{1}{R}+\frac{1}{s}+\frac{1}{r}\right)=\frac{R}{r}+\frac{r}{R}+(R+r)\left(\frac{s}{Rr}+\frac{1}{s}\right)+3=$$

$$=\frac{3R}{4r}+\left(\frac{R}{4r}+\frac{r}{R}\right)+(R+r)\left(\frac{25s}{27Rr}+\left(\frac{2s}{27Rr}+\frac{1}{s}\right)\right)+3\geq$$

$$\overset{\substack{Cosnita-Turtoiu\\ Euler\ and\ AM-GM}}{\geq}\frac{3}{2}+1+3\sqrt{\frac{Rr}{2}}\cdot\left(\frac{25}{27Rr}\cdot\sqrt{\frac{27Rr}{2}}+2\sqrt{\frac{2}{27Rr}}\right)+3=$$

$$=\frac{33+29\sqrt{3}}{6}$$

Equality holds iff ΔABC is equilateral.