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In any Δ ABC, the following relationship holds :

$$\frac{R}{s}+\frac{s}{r}+\frac{r}{R}\geq\frac{9+58\sqrt{3}}{18}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq}\frac{2}{3\sqrt{3}}+\frac{\sqrt{3(4Rr+r^2)}}{r}-\frac{29\sqrt{3}}{9}=$$

$$=\frac{2}{3\sqrt{3}}+\frac{3.\sqrt{4Rr+r^2}}{\sqrt{3}r}-\frac{29}{3\sqrt{3}}=\frac{9.\sqrt{4Rr+r^2}-27r}{3\sqrt{3}r}=\frac{3.\sqrt{4Rr+r^2}-9r}{\sqrt{3}r}\overset{?}{\geq}\frac{1}{2}-\frac{r}{R}$$

$$\Leftrightarrow\frac{9(4Rr+r^2)+81r^2-54r.\sqrt{4Rr+r^2}}{3r^2}\overset{?}{\geq}\frac{(R-2r)^2}{4R^2}$$

$$\Leftrightarrow\frac{3(4R+r)+27r}{r}-\frac{(R-2r)^2}{4R^2}\overset{?}{\geq}\frac{18.\sqrt{4Rr+r^2}}{r}$$

$$\Leftrightarrow\frac{(4R^2(12R+30r)-r(R-2r)^2)^2}{16R^4r^2}\overset{?}{\geq}\frac{324(4Rr+r^2)}{r^2}$$

$$\left(\begin{array}{c}\because\frac{3(4R+r)+27r}{r}\overset{\text{Euler}}{\geq}54>1\overset{\text{Euler}}{\geq}\frac{(R-2r)^2}{4R^2}\\ \Rightarrow 4R^2(12R+30r)-r(R-2r)^2>0\end{array}\right)$$

$$\Leftrightarrow 2304t^6-9312t^5+9361t^4+568t^3-936t^2-32t+16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(2304t^3+4512t^2+8785t+17566)+35136\right)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\ \therefore\frac{R}{s}+\frac{s}{r}-\frac{58\sqrt{3}}{18}\geq\frac{1}{2}-\frac{r}{R}$$

$$\Rightarrow\frac{R}{s}+\frac{s}{r}+\frac{r}{R}\geq\frac{9+58\sqrt{3}}{18}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{R}{s}+\frac{s}{r}+\frac{r}{R}=\left(\frac{R}{s}+\frac{2s}{27r}\right)+\left(\frac{r}{R}+\frac{\sqrt{3}s}{18r}+\frac{\sqrt{3}s}{18r}\right)+\frac{(25-3\sqrt{3})s}{27r}$$

$$\overset{AM-GM}{\geq}\ 2\sqrt{\frac{2R}{27r}}+3\sqrt[3]{\frac{s^2}{108Rr}}+\frac{(25-3\sqrt{3})s}{27r}$$

$$\overset{\substack{Cosnita-Turtoiu\\ Euler\ and\ Mitrinovic}}{\geq}\ 2\sqrt{\frac{4}{27}}+3\sqrt[3]{\frac{27}{108.2}}+\frac{(25-3\sqrt{3}).3\sqrt{3}}{27}=\frac{9+58\sqrt{3}}{18}.$$

Equality holds iff ΔABC is equilateral.