ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r}{s}+\frac{s}{R}+\frac{R}{r}\geq\frac{36+29\sqrt{3}}{18}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{r}{s}+\frac{s}{R}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen + Euler}}}{\geq}\frac{2r}{3\sqrt{3}R}+\frac{\sqrt{3(4Rr+r^2)}}{R}=\frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}$$

$$\therefore \text{it suffices to prove : }\frac{2\sqrt{3}r}{9R}+\frac{\sqrt{3(4Rr+r^2)}}{R}+\frac{R}{r}\geq\frac{36+29\sqrt{3}}{18}$$

$$\Leftrightarrow\frac{R}{r}-2\geq\sqrt{3}\left(\frac{29}{18}-\frac{2r}{9R}-\sqrt{\frac{4r}{R}+\frac{r^2}{R^2}}\right)$$

$$\Leftrightarrow t-2\overset{①}{\geq}\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\ \left(t=\frac{R}{r}\right)$$

$$\text{Let } f(t)=t-2-\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\ \forall\, t\geq 2 \text{ and then :}$$

$$f'(t)=1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{\sqrt{\frac{4}{t}+\frac{1}{t^2}}}\right)\overset{\frac{1}{t^2}>0}{>}1-\sqrt{3}\left(\frac{2}{9t^2}+\frac{\frac{2}{t^2}+\frac{1}{t^3}}{2\sqrt{\frac{1}{t}}}\right)$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{\frac{2}{m^4}+\frac{1}{m^6}}{\frac{2}{m}}\right)\ \left(m=\sqrt{t}\right)$$

$$=1-\sqrt{3}\left(\frac{2}{9m^4}+\frac{1}{m^3}+\frac{1}{2m^5}\right)\overset{m=\sqrt{\frac{R}{r}}\geq\sqrt{2}\text{ via Euler}}{\geq}1-\sqrt{3}\left(\frac{2}{9.4}+\frac{1}{2\sqrt{2}}+\frac{1}{8\sqrt{2}}\right)$$

$$\approx 0.1383>0\Rightarrow f'(t)>0\ \forall\, t\geq 2\Rightarrow f(t)\text{ is }\uparrow\text{ on }[2,\infty)\Rightarrow f(t)\geq f(2)=0\ \forall\, t\geq 2$$

$$\therefore t-2\geq\sqrt{3}\left(\frac{29}{18}-\frac{2}{9t}-\sqrt{\frac{4}{t}+\frac{1}{t^2}}\right)\Rightarrow ①\text{ is true }\therefore\frac{r}{s}+\frac{s}{R}+\frac{R}{r}\geq\frac{36+29\sqrt{3}}{18}$$

$\forall\,\Delta$ ABC," = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\frac{r}{s}+\frac{s}{R}+\frac{R}{r}=\left(\frac{r}{s}+\frac{2s}{27R}+\frac{\sqrt{3}R}{18r}\right)+\frac{25}{18}\left(\frac{s}{3R}+\frac{s}{3R}+\frac{\sqrt{3}R}{4r}\right)+\frac{(72-29\sqrt{3})R}{72r}$$

$$\overset{AM-GM}{\geq}\frac{3}{\sqrt[3]{81\sqrt{3}}}+\frac{25}{18}.3\sqrt[3]{\frac{s^2}{12\sqrt{3}Rr}}+\frac{(72-29\sqrt{3})R}{72r}$$

$$\overset{\substack{Cosnita\ Turtoiu\\ Euler}}{\geq}\frac{\sqrt{3}}{3}+\frac{25}{6}.\sqrt[3]{\frac{27}{12\sqrt{3}.2}}+\frac{(72-29\sqrt{3}).2}{72}=\frac{36+29\sqrt{3}}{18}.$$

Equality holds iff ΔABC is equilateral.