ROMANIAN MATHEMATICAL MAGAZINE

In any non − obtuse Δ ABC, the following relationship holds :

$$r_a \le 2R + r \le \frac{R}{r} \cdot h_a$$

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$$r_a \le 2R + r \Leftrightarrow \frac{rs}{s-a} - \frac{rs}{s} \le 2R \Leftrightarrow \frac{rs.(s-s+a)}{s(s-a)} \le 2R$$

$$\Leftrightarrow \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le 2R \Leftrightarrow \sin^2\frac{A}{2} \le \frac{1}{2} \Leftrightarrow \sin\frac{A}{2} \le \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \frac{A}{2} \le \frac{\pi}{4}\left(\because 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Leftrightarrow A \le \frac{\pi}{2} \to \text{true} \because \Delta\, ABC \text{ is non} - \text{obtuse} \therefore r_a \le 2R + r$$

$$\text{Again}, 2R + r \le \frac{R}{r}.h_a \Leftrightarrow R\left(\frac{2rs}{ra} - 2\right) \ge r \Leftrightarrow \frac{2(s-a)}{a} \ge \frac{r}{R}$$

$$\Leftrightarrow \frac{8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}} \ge 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \sin^2\frac{A}{2} \le \frac{1}{2} \Leftrightarrow \sin\frac{A}{2} \le \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \frac{A}{2} \le \frac{\pi}{4}\left(\because 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Leftrightarrow A \le \frac{\pi}{2} \to \text{true} \because \Delta\, ABC \text{ is non} - \text{obtuse}$$

$$\therefore \frac{R}{r}.h_a \ge 2R + r \text{ and so}, r_a \le 2R + r \le \frac{R}{r}.h_a\ \forall \text{ non} - \text{obtuse } \Delta\, ABC$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is right} - \text{angled at } \hat{A}\ (\text{QED})$$