ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a+r}{r_a-r}=\frac{8r_ar_br_c}{w_aw_bw_c}-2=3+\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}=\sum_{cyc}\frac{b+c}{a}$$

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1) $\sum_{cyc}\frac{r_a+r}{r_a-r}$

$$\sum_{cyc}\frac{r_a+r}{r_a-r}=\sum_{cyc}\frac{stan\frac{A}{2}+(s-a)tan\frac{A}{2}}{stan\frac{A}{2}-(s-a)tan\frac{A}{2}}=\sum_{cyc}\frac{(2s-a)tan\frac{A}{2}}{a.tan\frac{A}{2}}=\sum_{cyc}\frac{b+c}{a}\quad(1)$$

2) $\frac{8r_ar_br_c}{w_aw_bw_c}$

$$\frac{8r_ar_br_c}{w_aw_bw_c}-2=\frac{8sF}{\frac{2bc}{b+c}\cdot\frac{2ac}{a+c}\cdot\frac{2ba}{b+a}.cos\frac{A}{2}.cos\frac{B}{2}.cos\frac{C}{2}}-2=$$

$$=\frac{8sF(a+b)(b+c)(a+c)}{8(abc)^2.\frac{s}{4R}}-2=$$

$$=\frac{s\frac{abc}{4R}.(a+b)(b+c)(a+c)}{(abc)^2.\frac{s}{4R}}-2=\frac{(a+b)(b+c)(a+c)}{abc}-2=$$

$$=\frac{(a^2b+b^2c+c^2a+a^2c+c^2b+b^2a)+2ab}{abc}-2=$$

$$=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}=\sum_{cyc}\frac{b+c}{a}\quad(2)$$

3) $3+\sum_{cyc}\frac{h_a}{r_a}$

$$3+\sum_{cyc}\frac{h_a}{r_a}=3+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=3+\frac{h_ar_br_c+h_br_ar_c+h_cr_ar_b}{r_ar_cr_b}=$$

$$=3+\frac{\frac{2F}{a}s(s-a)+\frac{2F}{b}s(s-b)+\frac{2F}{c}s(s-c)}{sF}=3+\frac{2s-2a}{a}+\frac{2s-2b}{b}+\frac{2s-2c}{c}=$$

$$=\left(\frac{2s-2a}{a}+1\right)+\left(\frac{2s-2b}{b}+1\right)+\left(\frac{2s-2c}{c}+1\right)=$$

$$=\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}=\sum_{cyc}\frac{b+c}{a}\quad(3)$$

4) $\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}$

$$\sum_{cyc}\frac{s^2+r_br_c}{r_a(r_b+r_c)}=\sum_{cyc}\frac{s^2+r_br_c}{r_ar_b+r_ar_c}=\sum_{cyc}\frac{s^2+s(s-a)}{s(s-c)+s(s-b)}=$$

$$=\sum_{cyc}\frac{2s-a}{2s-(b+c)}=\sum_{cyc}\frac{b+c}{a}\quad(4)$$