ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_ah_b}{4rr_c}=\sum_{cyc}\frac{h_a+h_b}{2(r+r_c)}=\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}=3+\sum_{cyc}\frac{h_a-h_b}{2(r_a-r_b)}=\sum_{cyc}cos^2\frac{A}{2}$$

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Observation: $p=s$ –semiperimeter, $F=S-area$

1) $\sum_{cyc}\frac{h_ah_b}{4rr_c}$

$$\frac{h_ah_b}{4rr_c}=\frac{\frac{2S}{a}\cdot\frac{2S}{b}}{4r.ptan\frac{C}{2}}=\frac{4S^2}{4abrp.tan\frac{C}{2}}=\frac{abc.pr}{4Rabrp.tan\frac{C}{2}}=\frac{1}{4R}\cdot\frac{c}{tan\frac{C}{2}}=$$

$$=\frac{1}{4R}\cdot\frac{2RsinC}{tan\frac{C}{2}}=\frac{sin\frac{C}{2}.cos\frac{C}{2}}{tan\frac{C}{2}}=cos^2\frac{c}{2}\qquad\boxed{\sum_{cyc}\frac{h_ah_b}{4rr_c}=\sum_{cyc}cos^2\frac{C}{2}}$$

2) $\sum_{cyc}\frac{h_a+h_b}{2(r+r_c)}$

$$\frac{h_a+h_b}{2(r+r_c)}=\frac{\frac{2S}{a}+\frac{2S}{b}}{2\left(ptan\frac{C}{2}+(p-c)tan\frac{C}{2}\right)}=\frac{2S(a+b)}{2ab(a+b)tan\frac{C}{2}}=\frac{abc}{4Rab.tan\frac{C}{2}}=$$

$$=\frac{c}{4Rtan\frac{C}{2}}=\frac{2RsinC}{4Rtan\frac{C}{2}}=\frac{sin\frac{C}{2}.cos\frac{C}{2}}{tan\frac{C}{2}}=cos^2\frac{C}{2}\qquad\boxed{\sum_{cyc}\frac{h_a+h_b}{2(r+r_c)}=\sum_{cyc}cos^2\frac{C}{2}}$$

3) $\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}$

$$\frac{h_a+h_b}{4(r_a+r)}=\frac{\frac{2S}{a}+\frac{2S}{b}}{4(4R+r-r_c)}=\frac{2S(a+b)}{4ab\left(4R+(p-c)tan\frac{C}{2}-ptan\frac{C}{2}\right)}=$$

$$=\frac{2S(a+b)}{4ab\left(4R-ctan\frac{C}{2}\right)}=\frac{abc(a+b)}{2ab.4R.2R\left(2-sinCtan\frac{C}{2}\right)}=\frac{c(a+b)}{16R^2\left(2-2sin^2\frac{C}{2}\right)}=$$

$$=\frac{4R^2sinC(sinA+sinB)}{32R^2cos^2\frac{C}{2}}=\frac{16R^2.sin\frac{C}{2}.cos\frac{C}{2}}{32R^2cos^2\frac{C}{2}}.sin\frac{A+B}{2}.cos\frac{A-B}{2}=$$

$$=\frac{sin\frac{C}{2}.cos\frac{C}{2}.cos\frac{C}{2}}{2cos^2\frac{C}{2}}.cos\frac{A-B}{2}=\frac{1}{2}sin\frac{C}{2}\left(sin\frac{C+A-B}{2}+sin\frac{C+B-A}{2}\right)=\frac{1}{4}(cosB+cosA)$$

$$\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}=\frac{3}{2}+\sum_{cyc}\frac{1}{4}(cosB+cosA)=\frac{3}{2}+\frac{1}{2}(cosA+cosB+cosC)=$$

$$=\frac{3}{2}+\frac{1}{2}\left(-3+2\left(cos^2\frac{A}{2}+cos^2\frac{B}{2}+cos^2\frac{C}{2}\right)\right)=\frac{3}{2}+\sum_{cyc}cos^2\frac{C}{2}-\frac{3}{2}=\sum_{cyc}cos^2\frac{C}{2}$$

$$\boxed{\frac{3}{2}+\sum_{cyc}\frac{h_a+h_b}{4(r_a+r)}=\frac{3}{2}+\sum_{cyc}cos^2\frac{C}{2}-\frac{3}{2}=\sum_{cyc}cos^2\frac{C}{2}}$$

4) $3+\sum_{cyc}\frac{h_a-h_b}{2(r_a-r_b)}$

$$\frac{h_a-h_b}{2(r_a-r_b)}=\frac{\frac{2S}{a}-\frac{2S}{b}}{2\left(ptan\frac{A}{2}-ptan\frac{B}{2}\right)}=\frac{2S(b-a)}{2abp\left(tan\frac{A}{2}-tan\frac{B}{2}\right)}=\frac{abc.2R(sinB-sinA)}{4Rabp\frac{sin\frac{A-B}{2}}{cos\frac{A}{2}.cos\frac{B}{2}}}=$$

$$=\frac{c}{p}.\frac{sin\frac{B-A}{2}.cos\frac{A+B}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{sin\frac{A-B}{2}}=-\frac{2RsinC.sin\frac{C}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{p}=$$

$$=-\frac{4Rsin\frac{C}{2}.cos\frac{C}{2}.sin\frac{C}{2}.cos\frac{A}{2}.cos\frac{B}{2}}{p}=-\frac{4Rsin^2\frac{C}{2}\left(cos\frac{A}{2}.cos\frac{B}{2}.cos\frac{C}{2}\right)}{p}=-4R.\frac{sin^2\frac{C}{2}.\frac{P}{4R}}{p}=$$

$$=-sin^2\frac{C}{2}=-\left(1-cos^2\frac{C}{2}\right)=cos^2\frac{C}{2}-1$$

$$\boxed{3+\sum_{cyc}\frac{h_a-h_b}{2(r_a-r_b)}=\sum_{cyc}cos^2\frac{C}{2}}$$