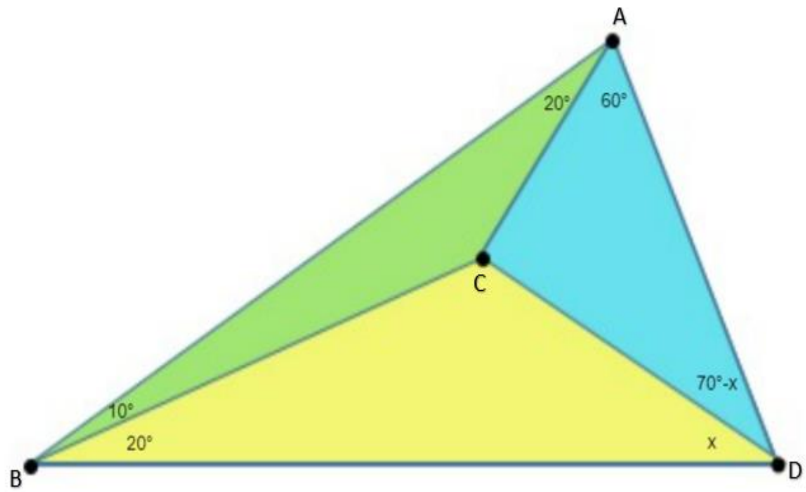
ROMANIAN MATHEMATICAL MAGAZINE

Prove that:



$$\frac{1}{[ABC]}+\frac{1}{[BCD]}=\frac{2}{[ADC]}$$

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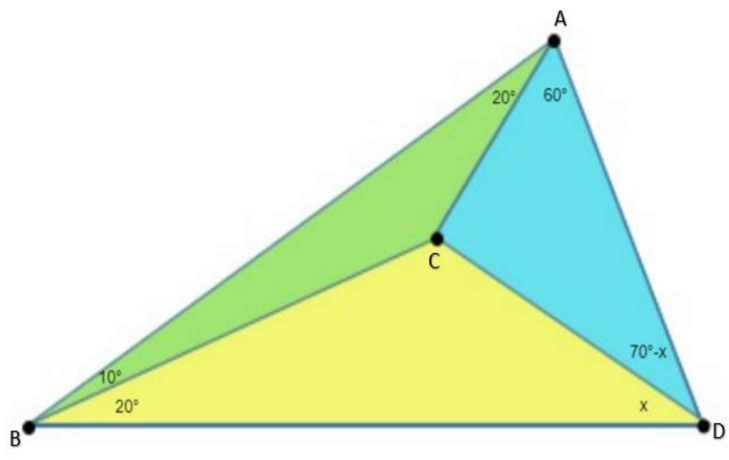
Solution by Mirsadix Muzefferov-Azerbaijan



By Ceva's theorem :

$$\sin 10^\circ \cdot \sin x \cdot \sin 60^\circ = \sin 20^\circ \cdot \sin(70^\circ - x) \cdot \sin 20^\circ$$

$$\sin 10^\circ \cdot \sin x \cdot \frac{\sqrt{3}}{2} = 2\sin 10^\circ \cdot \cos 10^\circ \cdot \sin(70^\circ - x) \cdot \sin 20^\circ$$

$$\frac{\sin x}{\sin(70^\circ - x)} = \frac{4\cos 10^\circ \cdot \sin 20^\circ \cdot \cos 20^\circ}{\sqrt{3}\cos 20^\circ} = \frac{2\sin 40^\circ \cdot \cos 10^\circ}{\sqrt{3}\cos 20^\circ}$$

$$= \frac{2\cos 10^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ}{\sqrt{3}\cos 20^\circ \cdot \cos 70^\circ} =$$

$$= \frac{2 \cdot \frac{\sqrt{3}}{8}}{\sqrt{3}\cos 20^\circ \cdot \sin 20^\circ} = \frac{1}{2\sin 40^\circ} = \frac{\sin 30^\circ}{\sin 40^\circ}$$

$$x = 30^\circ$$

$$Let\ CD = a\,.\,In\ \Delta BCD \Rightarrow \frac{BC}{\sin 30^\circ} = \frac{CD}{\sin 20^\circ} \Rightarrow BC = \frac{a}{2\sin 20^\circ} \quad (1)$$

$$In\ \Delta ACD \Rightarrow \frac{AC}{\sin 40^\circ} = \frac{CD}{\sin 60^\circ} \Rightarrow AC = \frac{2a\sin 40^\circ}{\sqrt{3}} \quad (2)$$

$$[ABC] \overset{(1)}{\underset{(2)}{=}} \frac{1}{2} \cdot BC \cdot AC \cdot \sin 30^\circ = \frac{1}{2} \cdot \frac{a}{2\sin 20^\circ} \cdot \frac{2a\sin 40^\circ}{\sqrt{3}} \cdot \frac{1}{2} = \frac{a^2 \sin 40^\circ}{4\sqrt{3}\sin 20^\circ} \quad (3)$$

$$[BCD] = \frac{1}{2} \cdot BC \cdot CD \cdot \sin 50^\circ \overset{(1)}{=} \frac{1}{2} \cdot \frac{a}{2\sin 20^\circ} \cdot a \cdot \sin 50^\circ = \frac{a^2 \sin 50^\circ}{4\sin 20^\circ} \quad (4)$$

$$[ACD] = \frac{1}{2} \cdot AC \cdot CD \cdot \sin 80^\circ \overset{(2)}{=} \frac{1}{2} \cdot \frac{2a\sin 40^\circ}{\sqrt{3}} \cdot a \cdot \sin 80^\circ = \frac{a^2 \sin 40^\circ \cos 10^\circ}{\sqrt{3}} \quad (5)$$

$$\frac{\frac{1}{[ABC]} + \frac{1}{[BCD]}}{\frac{1}{[ADC]}} \overset{(3)(4)}{\underset{(5)}{=}} \frac{\frac{4\sqrt{3}\sin 20^\circ}{a^2 \sin 40^\circ} + \frac{4\sin 20^\circ}{a^2 \sin 50^\circ}}{\frac{\sqrt{3}}{a^2 \sin 40^\circ \cos 10^\circ}} =$$

$$= \frac{4\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\sqrt{3}} \cdot \frac{2\left(\sqrt{3}\cos 40^\circ + \sin 40^\circ\right)}{2\sin 40^\circ \cos 40^\circ} =$$

$$= \frac{4\sin 20^\circ \sin 40^\circ}{\sqrt{3}} \cdot 4 \cdot \left(\frac{\sqrt{3}}{2}\cos 40^\circ + \frac{1}{2}\sin 40^\circ\right) =$$

$$= \frac{16}{\sqrt{3}} \cdot \sin 20^\circ \sin 40^\circ \left(\sin 60^\circ \cos 40^\circ + \cos 60^\circ \sin 40^\circ\right) =$$

$$= \frac{16}{\sqrt{3}} \cdot \sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{16}{\sqrt{3}} \cdot \frac{\sqrt{3}}{8} = 2$$