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ON AN INEQUALITY AND IT`S COROLLARIES

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SUMMARY

In the present article it is given the proofs of several inequalities (T1, T2, T3) which are useful In proving of inequalities. The particular cases are discussed and an application to the solving of concrete examples is given.

There is no a universal method for proving of inequalities. However, one of the well known methods is the application of known inequalities.

Theorem 1. if $a_{i,}\ b_i\ \ (i=1,2,...,n)$ – are positive numbers, prove the inequalities.

a) $$\frac{a_1^{\ p}}{b_1^{\ p-1}}+\frac{a_2^{\ p}}{b_2^{\ p-1}}+\cdots+\frac{a_n^{\ p}}{b_n^{\ p-1}}\ge\frac{(a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}},\quad p\notin(0;1)\qquad \textbf{(M)}$$

b) $$\frac{a_1^{\ p}}{b_1^{\ p-1}}+\frac{a_2^{\ p}}{b_2^{\ p-1}}+\cdots+\frac{a_n^{\ p}}{b_n^{\ p-1}}\le\frac{(a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}},\quad p\in[0;1]\qquad \textbf{(F)}$$

Proof. Let's look at the function:. $\varphi(y)=\frac{x^\alpha}{y^{\alpha-1}}-\alpha x+(\alpha-1)y; x\ >0, y\ >0$

find the derivative of the function $\varphi(y)$: $\varphi^{/}(y)=(\alpha-1)\left(1-\left(\frac{x}{y}\right)^\alpha\right)$

Let's look at two cases

1. Suppose, $\alpha\in(-\infty;0)\cup(1;\infty)$. Then will be $y<x$, will be $\varphi^{/}(y)<0; y\ =x$, will be $\varphi^{/}(y)=0;\ \ y>x$, will be $\varphi^{/}(y)>0$. It means, $y=x$ point is minimum point of $\varphi(y)$ function, that is $\varphi(y)\ge\varphi(x)$ .Then

$$\frac{x^\alpha}{y^{\alpha-1}}-\alpha x+(\alpha-1)y\ge\frac{x^\alpha}{x^{\alpha-1}}-\alpha x+\alpha x-x=0.$$

If $\alpha=0$ and $\alpha=1$, and since $\varphi(y)=0$, for any $\alpha\in(-\infty;0]\cup[1;\infty)$, that is for $\alpha\notin(0;1)$

$$\frac{x^\alpha}{y^{\alpha-1}}\ge\alpha x-(\alpha-1)y\qquad \textbf{(1)}\qquad \text{inequality is correct}$$

2. Suppose $\alpha\in(0;1)$. Then will be $y<x$, will be $\varphi^{/}(y)>0;\ \ y=x$, will be $\varphi^{/}(y)=0;\ \ y>x$, will be $\varphi^{/}(y)<0$
It means, $y=x$ point is maximum point of $\varphi(y)$ function, that is $\varphi(y)\le\varphi(x)$ .Then

$$\frac{x^\alpha}{y^{\alpha-1}}-\alpha x+(\alpha-1)y\le\frac{x^\alpha}{x^{\alpha-1}}-\alpha x+\alpha x-x=0.$$

If $\alpha=0$ and $\alpha=1$ and since $\varphi(y)=0$, for any $\alpha\in[0;1]$

$$\frac{x^\alpha}{y^{\alpha-1}}\le\alpha x-(\alpha-1)y\qquad \textbf{(2)}\qquad \text{inequality is correct}$$

.

(M) let's show that the inequality is true. For any $p\notin(0;1)$ and $k>0$ number

Let's apply inequality (1) to any of item in expression $\frac{(ka_1)^p}{b_1^{\ p-1}}+\frac{(ka_2)^p}{b_2^{\ p-1}}+\cdots\cdots+\frac{(ka_n)^p}{b_n^{\ p-1}}$. Then

$$\frac{(ka_1)^p}{b_1^{\,p-1}}+\frac{(ka_2)^p}{b_2^{\,p-1}}+\cdots+\frac{(ka_n)^p}{b_n^{\,p-1}}\ge kp(a_1+a_2+\cdots+a_n)-(p-1)(b_1+b_2+\cdots+b_n).$$

If we divide both sides of inequality to k^p

$$\frac{a_1^{\,p}}{b_1^{\,p-1}}+\frac{a_2^{\,p}}{b_2^{\,p-1}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,p-1}}\ge\frac{p}{k^{p-1}}(a_1+a_2+\cdots+a_n)-\frac{p-1}{k^p}(b_1+b_2+\cdots+b_n)\qquad (*)\ .$$

Since $(*)$ inequality is correct for $k>0$ when $k=\dfrac{b_1+b_2+\cdots+b_n}{a_1+a_2+\cdots+a_n}$ it is correct again.

If we take into account this value of k in $(*)$:

$$\frac{a_1^{\,p}}{b_2^{\,p-1}}+\frac{a_2^{\,p}}{b_2^{\,p-1}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,p-1}}\ge\frac{p(a_1+a_2+\cdots+a_n)}{\left(\dfrac{b_1+b_2+\cdots+b_n}{a_1+a_2+\cdots+a_n}\right)^{p-1}}-\frac{(p-1)(b_1+b_2+\cdots+b_n)}{\left(\dfrac{b_1+b_2+\cdots+b_n}{a_1+b_2+\cdots+a_n}\right)^{p}}=$$

$$=\frac{p(a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}}-\frac{(p-1)\ (a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}}=\frac{(a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}}\ .$$

The inequality **(M)** is proved.

Let's show the correctness of **(F)** inequality. For any $p\in[0;1]$ and $k>0$ number

Let's apply inequality **(2)** to any of item in expression $\dfrac{(ka_1)^p}{b_1^{\,p-1}}+\dfrac{(ka_2)^p}{b_2^{\,p-1}}+\cdots\cdots+\dfrac{(ka_n)^p}{b_n^{\,p-1}}$

Then will be

$$\frac{(ka_1)^p}{b_1^{\,p-1}}+\frac{(ka_2)^p}{b_2^{\,p-1}}+\cdots+\frac{(ka_n)^p}{b_n^{\,p-1}}\le kp(a_1+a_2+\cdots+a_n)-(p-1)(b_1+b_2+\cdots+b_n).$$

If we divide both sides of inequality to k^p, and accept as $k=\dfrac{b_1+b_2+\cdots+b_n}{a_1+a_2+\cdots+a_n}$,

$$\frac{a_1^{\,p}}{b_1^{\,p-1}}+\frac{a_2^{\,p}}{b_2^{\,p-1}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,p-1}}\le\frac{(a_1+a_2+\cdots+a_n)^p}{(b_1+b_2+\cdots+b_n)^{p-1}},p\ \in[0;1]$$ then inequality will be proved.

The condition of equality in **(M)** and **(F)** inequalities is possible of $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\cdots=\dfrac{a_n}{b_n}$.

Result 1. if $a_i>0\ \ (i=1,2,\cdots,n)$, prove the inequality.

$$a_1^p+a_2^p+\cdots+a_n^p\ge\frac{1}{n^{p-1}}(a_1+a_2+\cdots+a_n)^p;\quad p\notin(0;1)\qquad (M_1)$$

Proof. in **(M)** inequality if $b_1=b_2=\cdots=b_n=1$, then the proof of inequality is clear.

Result 2. if $a_i>0\ \ (i=1,2,\cdots,n)$ prove the inequality.

$$a_1^p+a_2^p+\cdots+a_n^p\le\frac{1}{n^{p-1}}(a_1+a_2+\cdots+a_n)^p;\quad p\in[0;1]\qquad (F_1)$$

Proof. in **(F)** inequality if $b_1=b_2=\cdots=b_n=1$ then the proof of inequality is clear.

Theorem 2. if $a_{i,}\ b_i\ (i=1,2,...,n)$ are positive numbers, then prove the inequality.

$$\frac{a_1^{\,p}}{b_1^{\,q}}+\frac{a_2^{\,p}}{b_2^{\,q}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,q}}\ge\frac{(a_1+a_2+\cdots+a_n)^p}{n^{p-(q+1)}(b_1+b_2+\cdots+b_n)^q}\quad;\quad 0<q\le p-1\qquad \textbf{(A)}$$

Proof. Since $p>1$, lets apply **(M)** inequality.

$$\frac{a_1^{\,p}}{b_1^{\,q}}+\frac{a_2^{\,p}}{b_2^{\,q}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,q}}=\frac{a_1^p}{\left(b_1^{\frac{q}{p-1}}\right)^{p-1}}+\frac{a_2^p}{\left(b_2^{\frac{q}{p-1}}\right)^{p-1}}+\cdots+\frac{a_n^p}{\left(b_n^{\frac{q}{p-1}}\right)^{p-1}}\geq\frac{(a_1+a_2+\cdots+a_n)^p}{\left(b_1^{\frac{q}{p-1}}+b_2^{\frac{q}{p-1}}+\cdots+b_n^{\frac{q}{p-1}}\right)^{p-1}}\quad\textbf{(1)}$$

According to condition as $0<q\leq p-1$, will be $0<\frac{q}{p-1}\leq 1$. According to (F_1) inequality

$$b_1^{\frac{q}{p-1}}+b_2^{\frac{q}{p-1}}+\cdots+b_n^{\frac{q}{p-1}}\leq\frac{1}{n^{\frac{q}{p-1}-1}}(b_1+b_2+\cdots+b_n)^{\frac{q}{p-1}}=n^{\frac{p-(q+1)}{p-1}}(b_1+b_2+\cdots+b_n)^{\frac{q}{p-1}}\qquad\textbf{(2)}$$

If we take (2) into account in (1), then the proof of given inequality is clear. The equality case is possible when $a_1=a_2=\cdots=a_n$; $b_1=b_2=\cdots=b_n$.

Special case: if $q=1$,

$$\frac{a_1^p}{b_1}+\frac{a_2^p}{b_2}+\cdots+\frac{a_n^p}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^p}{n^{p-2}(b_1+b_2+\cdots+b_n)}\ ;\ p\geq 2\qquad (A_1)$$

Theorem 3. If $a_{i,}\ b_i\ (i=1,2,...,n)$ are positive numbers, prove the inequality.

$$\frac{a_1^{\,p}}{b_1^{\,q}}+\frac{a_2^{\,p}}{b_2^{\,q}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,q}}\leq\frac{(a_1+a_2+\cdots+a_n)^p}{n^{p-(q+1)}(b_1+b_2+\cdots+b_n)^q}\ ;\ 0<p<1\ ;\ q\leq p-1\qquad\textbf{(B)}$$

Proof. Since $0<p<1$, let's apply **(F)** inequality.

$$\frac{a_1^{\,p}}{b_1^{\,q}}+\frac{a_2^{\,p}}{b_2^{\,q}}+\cdots+\frac{a_n^{\,p}}{b_n^{\,q}}=\frac{a_1^p}{\left(b_1^{\frac{q}{p-1}}\right)^{p-1}}+\frac{a_2^p}{\left(b_2^{\frac{q}{p-1}}\right)^{p-1}}+\cdots+\frac{a_n^p}{\left(b_n^{\frac{q}{p-1}}\right)^{p-1}}\leq\frac{(a_1+a_2+\cdots+a_n)^p}{\left(b_1^{\frac{q}{p-1}}+b_2^{\frac{q}{p-1}}+\cdots+b_n^{\frac{q}{p-1}}\right)^{p-1}}\quad\textbf{(1)}$$

According to condition since $0<p<1$ and $q\leq p-1$, it will be $\frac{q}{p-1}\geq 1$. According to (M_1) inequality

$$b_1^{\frac{q}{p-1}}+b_2^{\frac{q}{p-1}}+\cdots+b_n^{\frac{q}{p-1}}\geq\frac{1}{n^{\frac{q}{p-1}-1}}(b_1+b_2+\cdots+b_n)^{\frac{q}{p-1}}=n^{\frac{p-(q+1)}{p-1}}(b_1+b_2+\cdots+b_n)^{\frac{q}{p-1}}\qquad\textbf{(2)}$$

If take into account **(2)** in **(1),** then the proof of given inequality is clear. The equality case is possible when $a_1=a_2=\cdots=a_n$; $b_1=b_2=\cdots=b_n$.

Special case: if $q=-1$,

$$a_1^pb_1+a_2^pb_2+\cdots+a_n^pb_n\leq\frac{1}{n^p}(a_1+a_2+\cdots+a_n)^p(b_1+b_2+\cdots+b_n)\ ;\ (0<p<1)\qquad (B_1)$$

Example 1. a**,** b**,** c are positive numbers. Prove that

$$4a^3+9b^3+36c^3\geq(a+b+c)^3$$

Solution. In (M) inequality $p=3$, if $n=3$, then

$$4a^3+9b^3+36c^3=\frac{a^3}{\left(\frac{1}{2}\right)^2}+\frac{b^3}{\left(\frac{1}{3}\right)^2}+\frac{c^3}{\left(\frac{1}{6}\right)^2}\geq\frac{(a+b+c)^3}{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)^2}=(a+b+c)^3$$

Equality case is obtained when $a=3$, $b=2$, $c=1$.

Example 2. a, b, c are positive numbers. Prove that

$$a\sqrt{\frac{a}{b+c}}+b\sqrt{\frac{b}{c+a}}+c\sqrt{\frac{c}{a+b}}\geq\frac{1}{\sqrt{2}}(a+b+c)$$

Solution. In (M) inequality $p=\frac{3}{2}$, if $n=3$, then

$$a\sqrt{\frac{a}{b+c}}+b\sqrt{\frac{b}{c+a}}+c\sqrt{\frac{c}{a+b}}=\frac{a^{\frac{3}{2}}}{(b+c)^{\frac{1}{2}}}+\frac{b^{\frac{3}{2}}}{(c+a)^{\frac{1}{2}}}+$$

$$+\frac{c^{\frac{3}{2}}}{(a+b)^{\frac{1}{2}}}\geq\frac{(a+b+c)^{\frac{3}{2}}}{(2(a+b+c))^{\frac{1}{2}}}=\frac{1}{\sqrt{2}}(a+b+c)$$

Equality case is obtained when $a=b=c$

Example 3. Sum of a,b,c positive numbers is equal to 3. Prove that

$$\sqrt[3]{a^2}\cdot\sqrt[5]{b^3}+\sqrt[3]{b^2}\cdot\sqrt[5]{c^3}+\sqrt[3]{c^2}\cdot\sqrt[5]{a^3}\leq 3$$

Solution. According to (B) inequality $\left(p=\frac{2}{3},q=-\frac{3}{5},n=3\right)$

$$\sqrt[3]{a^2}\cdot\sqrt[5]{b^3}+\sqrt[3]{b^2}\cdot\sqrt[5]{c^3}+\sqrt[3]{c^2}\cdot\sqrt[5]{a^3}=\frac{a^{\frac{2}{3}}}{b^{-\frac{3}{5}}}+\frac{b^{\frac{2}{3}}}{c^{-\frac{3}{5}}}+\frac{c^{\frac{2}{3}}}{a^{-\frac{3}{5}}}\leq$$

$$\leq\frac{(a+b+c)^{\frac{2}{3}}}{3^{\frac{2}{3}-\left(-\frac{3}{5}+1\right)}\cdot(a+b+c)^{-\frac{3}{5}}}=\frac{1}{3^{\frac{4}{15}}}\cdot\frac{3^{\frac{2}{3}}}{3^{-\frac{3}{5}}}=3$$

Equality case is obtained when $a=b=c=1$.

Example 4. if $a>0$, $b>0$, $c>0$, prove an inequality.

$$\frac{b^2c^2}{a^3(b+c)}+\frac{a^2c^2}{b^3(a+c)}+\frac{a^2b^2}{c^3(a+b)}\geq\frac{3}{2}$$

Solution. If we mark $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{t}$ an inequality will be in a form like

$$\frac{x^3}{yt(y+t)}+\frac{y^3}{xt(x+t)}+\frac{t^3}{xy(x+y)}\geq\frac{3}{2}$$

Let's multiply both sides of inequality to xyt.

$$\frac{x^4}{y+t}+\frac{y^4}{x+t}+\frac{t^4}{x+y}\geq\frac{3}{2}xyt \qquad \textbf{(1)}$$

Let's show correctness of **(1).** According to (A_1) inequality

$$\frac{x^4}{y+t}+\frac{y^4}{x+t}+\frac{t^4}{x+y}\geq\frac{(x+y+t)^4}{3^2(2(x+y+t))}=\frac{(x+y+t)^3}{18}\geq\frac{\left(3\cdot\sqrt[3]{xyt}\right)^3}{18}=\frac{27xyt}{18}=\frac{3}{2}xyt$$

Equality case is obtained when $a=b=c$

Example 5. In right-angled triangle if a, b are cathetus and c - hypotenuse, then prove the inequality.

$$a^n+b^n\geq\left(\frac{1}{\sqrt{2}}\right)^{n-2}c^n;\ n\in N, n\geq 3$$

Solution. In right-angled triangle the inequality $a+b\leq c\sqrt{2}$ is correct. Let's see at cases of odd and even n.

1. if $n=2k+1$; $k\in N$, let's prove $a^{2k+1}+b^{2k+1}\geq\left(\frac{1}{\sqrt{2}}\right)^{2k-1}c^{2k+1}$ inequality.

Let's apply (A_1) inequality. $a^{2k+1}+b^{2k+1}=\frac{(a^2)^{k+1}}{a}+\frac{(b^2)^{k+1}}{b}\geq$

$$\geq\frac{(a^2+b^2)^{k+1}}{2^{k+1-2}(a+b)}=\frac{(c^2)^{k+1}}{2^{k-1}(a+b)}\geq\frac{c^{2k+2}}{\left(\sqrt{2}\right)^{2k-2}\sqrt{2}\cdot c}=\left(\frac{1}{\sqrt{2}}\right)^{2k-1}c^{2k+1}$$

2. if $n=2k$; $k\in N$, $k\geq 2$, let's prove $a^{2k}+b^{2k}\geq\left(\frac{1}{\sqrt{2}}\right)^{2k-2}c^{2k}$ inequality.

(M) Let's apply inequality $a^{2k}+b^{2k}=\frac{(a^2)^k}{1^{k-1}}+\frac{(b^2)^k}{1^{k-1}}\geq\frac{(a^2+b^2)^k}{(1+1)^{k-1}}=\frac{c^{2k}}{2^{k-1}}=\left(\frac{1}{\sqrt{2}}\right)^{2k-2}c^{2k}$

Thus, the given inequality is correct for any natural number $n(n>2)$. If equality case is $a=b$, it is correct

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